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Representations of the ultrahyperbolic BMS group \mathcal{HB} .

III. Determination of the representations induced from finite little groups

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Abstract

The ordinary Bondi–Metzner–Sachs (BMS) group B is the common asymptotic symmetry group of all asymptotically flat Lorentzian space-times. As such, B is the best candidate for the universal symmetry group of General Relativity. However, in studying quantum gravity, space–times with signatures other than the usual Lorentzian one, and complex space–times, are frequently considered. Generalisations of B appropriate to these other signatures have been defined earlier. In particular, \mathcal{HB} , a variant of BMS group appropriate to the ultrahyperbolic signature $(+, +, -, -)$, has been defined in a previous paper where it was shown that all the strongly continuous unitary irreducible representations (IRs) of \mathcal{HB} can be obtained with the Wigner–Mackey’s inducing method and that all the little groups of \mathcal{HB} are compact. Here we describe in detail all the finite little groups of \mathcal{HB} and we find the IRs of \mathcal{HB} induced by them.

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1 Introduction

The best candidate for the universal symmetry group of General Relativity (G.R), in any signature, is the so called Bondi–Metzner–Sachs (BMS) group B . These groups have recently been described [1] for all possible signatures and all possible complex versions of GR as well.

In earlier papers [1, 2, 3, 4, 5] it has been argued that the IRs of the BMS group and of its generalizations in complex space–times as well as in space–times with Euclidean or Ultrahyperbolic signature are what really lie behind the full description of (unconstrained) moduli spaces of gravitational instantons. Kronheimer [6, 7] has given a description of these instanton moduli spaces for *Euclidean* instantons. However, his description only partially describes the moduli spaces, since it still involves *constraints*. Kronheimer does not solve the constraint equations, but it has been argued [1, 5] that IRs of BMS group (in the relevant signature) give an *unconstrained* description of these same moduli spaces.

The original BMS group B was discovered by Bondi, Metzner and Van der Burg [8] for asymptotically flat space–times which were axisymmetric, and by Sachs [9] for general asymptotically flat space–times, in the usual Lorentzian signature. The group \mathcal{HB} is a different generalised BMS group, namely one appropriate to the ‘ultrahyperbolic’ signature, and asymptotic flatness in null directions introduced in [10].

Recall that the ultrahyperbolic version of Minkowski space is the vector space R^4 of row vectors with 4 real components, with scalar product defined as follows. Let $x, y \in R^4$ have components x^μ and y^μ respectively, where $\mu = 0, 1, 2, 3$. Define the scalar product $x.y$ between x and y by

$$x.y = x^0 y^0 + x^2 y^2 - x^1 y^1 - x^3 y^3. \quad (1.1)$$

Then the ultrahyperbolic version of Minkowski space, sometimes written $R^{2,2}$, is just R^4 with this scalar product.

In [10] it was shown that

Theorem 1 *The group \mathcal{HB} can be realised as*

$$\mathcal{HB} = L^2(\mathcal{P}, \lambda, R) \mathbin{\mathbb{S}}_T G^2 \quad (1.2)$$

with semi–direct product specified by

$$(T(g, h)\alpha)(x, y) = k_g(x)s_g(x)k_h(w)s_h(w)\alpha(xg, yh), \quad (1.3)$$

where $\alpha \in L^2(\mathcal{P}, \lambda, R)$, the separable Hilbert space of real-valued functions defined on \mathcal{P} , and $(x, y) \in \mathcal{P}$. For ease of notation, we write \mathcal{P} for the torus $T \simeq P_1(R) \times P_1(R)$, $P_1(R)$ is the one-dimensional real projective space, and \mathcal{G} for $G \times G$, $G = SL(2, R)$. In analogy to B , it is natural to choose a measure λ on \mathcal{P} which is invariant under the maximal compact subgroup $SO(2) \times SO(2)$ of \mathcal{G} .

Moreover, if $g \in G$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad (1.4)$$

then the components x_1, x_2 of $\mathbf{x} \in R^2$ transform linearly, so that the ratio $x = x_1/x_2$ transforms fraction linearly. Writing xg for the transformed ratio,

$$xg = \frac{(\mathbf{x}g)_1}{(\mathbf{x}g)_2} = \frac{x_1a + x_2c}{x_1b + x_2d} = \frac{xa + c}{xb + d}. \quad (1.5)$$

The factors $k_g(x)$ and $s_g(x)$ on the right hand side of (1.3) are defined by

$$k_g(x) = \left\{ \frac{(xb + d)^2 + (xa + c)^2}{1 + x^2} \right\}^{\frac{1}{2}}, \quad (1.6)$$

$$s_g(x) = \frac{xb + d}{|xb + d|}, \quad (1.7)$$

with similar formulae for yh , $k_h(y)$ and $s_h(y)$.

It is well known that the topological dual of a Hilbert space can be identified with the Hilbert space itself, so that we have $L^{2'}(\mathcal{P}, \lambda, R) \simeq L^2(\mathcal{P}, \lambda, R)$. In fact, given a continuous linear functional $\phi \in L^{2'}(\mathcal{P}, \lambda, R)$, we can write, for $\alpha \in L^2(\mathcal{P}, \lambda, R)$

$$(\phi, \alpha) = \langle \phi, \alpha \rangle \quad (1.8)$$

where the function $\phi \in L^2(\mathcal{P}, \lambda, R)$ on the right is uniquely determined by (and denoted by the same symbol as) the linear functional $\phi \in L^{2'}(\mathcal{P}, \lambda, R)$ on the left. The representation theory of \mathcal{HB} is governed [10] by the dual action T' of \mathcal{G} on the topological dual $L^{2'}(\mathcal{P}, \lambda, R)$ of $L^2(\mathcal{P}, \lambda, R)$. The dual action T' is defined by:

$$\langle T'(g, h)\phi, \alpha \rangle = \langle \phi, T(g^{-1}, h^{-1})\alpha \rangle. \quad (1.9)$$

A short calculation gives

$$(T'(g, h)\phi)(x, y) = k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y)\phi(xg, yh). \quad (1.10)$$

Now, this action T' of \mathcal{G} on $L^2(\mathcal{P}, \lambda, R)$, given explicitly above, is like the action T of \mathcal{G} on $L^2(\mathcal{P}, \lambda, R)$, continuous. The ‘little group’ L_ϕ of any $\phi \in L^2(\mathcal{P}, \lambda, R)$ is the stabilizer

$$L_\phi = \{(g, h) \in \mathcal{G} \mid T'(g, h)\phi = \phi\}. \quad (1.11)$$

By continuity, $L_\phi \subset \mathcal{G}$ is a closed subgroup.

Attention is confined to measures on $L^2(\mathcal{P}, \lambda, R)$ which are concentrated on single orbits of the \mathcal{G} –action T' . These measures give rise to IRs of \mathcal{HB} which are induced in a sense generalising [11] Mackey’s [12, 13, 14, 15, 16, 17]. In fact in [10] it was shown that *all* IRs of the \mathcal{HB} with the Hilbert topology are derivable by the inducing construction. The inducing construction is realized as follows. Let $\mathcal{O} \subset L^2(\mathcal{P}, \lambda, R)$ be any orbit of the dual action T' of \mathcal{G} on $L^2(\mathcal{P}, \lambda, R)$. There is a natural homomorphism $\mathcal{O} = \mathcal{G}\phi_o \simeq \mathcal{G}/L_{\phi_o}$ where L_{ϕ_o} is the ‘little group’ of the point $\phi_o \in \mathcal{O}$. Let U be a continuous irreducible unitary representation of L_{ϕ_o} on a Hilbert space D . Every coset space \mathcal{G}/L_{ϕ_o} can be equipped with a unique class of measures which are quasi–invariant under the action T of G^2 . Let μ be any one of these. Let $D_\mu = L^2(\mathcal{G}/L_{\phi_o}, \mu, D)$ be the Hilbert space of functions $f : \mathcal{G}/L_{\phi_o} \rightarrow D$ which are square integrable with respect to μ . From a given ϕ_o and any continuous irreducible unitary representation U of L_{ϕ_o} on a Hilbert space D a continuous irreducible unitary representation of \mathcal{HB} on D_μ can be constructed. The representation is said to be induced from U and ϕ_o . Different points of an orbit $\mathcal{G}\phi$ have conjugate little groups and give rise to equivalent representations of \mathcal{HB} .

To conclude, every irreducible representation of \mathcal{HB} is obtained by the inducing construction for each $\phi_o \in L^2(\mathcal{P}, \lambda, R)$ and each irreducible representation U of L_{ϕ_o} . All the little groups L_{ϕ_o} of \mathcal{HB} are compact and they up to conjugation subgroups of $\text{SO}(2) \times \text{SO}(2)$. They include groups which are finite as well as groups which are infinite, both connected and not–connected [2]. Therefore the construction of the IRs of \mathcal{HB} involves at the first instance the classification of all the subgroups of $\text{SO}(2) \times \text{SO}(2)$.

The infinite not–connected subgroups of $\text{SO}(2) \times \text{SO}(2)$ were given in [2] and the infinite connected subgroups of $\text{SO}(2) \times \text{SO}(2)$ were given in [18]. The IRs of \mathcal{HB} induced from all the infinite little groups, both connected and non–connected, were constructed in [18].

The problem of constructing the IRs of \mathcal{HB} induced from *finite* little groups reduces to a seemingly very simple task [2]; that of classifying all subgroups of the Cartesian product group $C_n \times C_m$, where C_r is the cyclic group of order r , r being finite. Surprisingly, this task is less simple than it may appear at first sight. It turns out [3] that the solution

is constructed from the ‘fundamental cases’ $n = p^a$, $m = p^b$, (n, m are powers of the same prime), via the prime decomposition of m and n . The classification of all the subgroups of $C_n \times C_m$, was given in [3]. Classifying the subgroups of $C_n \times C_m$ is one thing, constructing the IRs of \mathcal{HB} induced by the finite little groups is quite another. In this paper we fill this gap by isolating the finite little groups and by explicitly constructing the IRs of \mathcal{HB} induced by them.

The paper is organised as follows: In Section 2 we prove that the elementary domains for the actions of $G = C_n \times C_m$ and of any of its subgroups F on the torus \mathcal{P} are related in a very simple fashion: the elementary domain for the action of F on \mathcal{P} can be obtained by letting a set of representatives of the left cosets of F in the coset space G/F act on an elementary domain E for the action of G on \mathcal{P} . In Section 3 we use this result to find the Hilbert spaces $\mathcal{H}(F)$ of all invariant vectors for each of the finite subgroups F of G . In Section 4 we find all the finite potential little groups. In Section 5 we prove that all the finite potential little groups are actual. In Section 6 we describe explicitly all the finite little groups. In Section 7 we give the form of the IRs of \mathcal{HB} induced by the IRs of the finite little groups and the corresponding invariant characters. Finally, in Section 8 we make some remarks about the IRs of \mathcal{HB} constructed in Section 7 by the inducing method.

2 Elementary regions for Finite Groups

The representation theory of \mathcal{HB} is governed by the dual action T' of \mathcal{G} on $L^{2'}(\mathcal{P}, \lambda, R)$ given by Eq. (1.10)

$$(T'(g, h)\phi)(x, y) = k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y)\phi(xg, yh).$$

We have an action

$$(x, y) \longmapsto (xg, yh) \tag{2.1}$$

of \mathcal{G} on the torus $T \simeq P_1(R) \times P_1(R)$. We need at this point to recall some results regarding group actions. Recall that if the set \mathcal{X} is a G -space then there is a natural bijective correspondence between the set of actions of G on \mathcal{X} and the set of homomorphisms from G into $\text{Aut}(\mathcal{X})$, where $\text{Aut}(\mathcal{X})$ is the group of automorphisms of the set \mathcal{X} . Indeed, let the map

$$\mathcal{X} \times G \mapsto \mathcal{X}, \tag{2.2}$$

from $\mathcal{X} \times G$ to \mathcal{X} be a right action of G on the set \mathcal{X} , with the image of (x, g) being denoted by xg . Then the map (2.2) satisfies the following conditions:

- $xe = x$ for every $x \in \mathcal{X}$.
- $x(g_1g_2) = (xg_1)g_2$ for every $g_1, g_2 \in G$ and $x \in \mathcal{X}$.

Now for each $g \in G$ define the map

$$\begin{aligned} s_g &: \mathcal{X} \rightarrow \mathcal{X} \\ s_g(x) &= xg \text{ for } x \in \mathcal{X}. \end{aligned} \quad (2.3)$$

One can easily show (see e.g [19], p.28) that s_g is a member of the group $\text{Aut}(\mathcal{X})$. Furthermore, the second condition in the definition of a group action ensures that we have $s_{g_1g_2} = s_{g_1} \circ s_{g_2}$ for any $g_1, g_2 \in G$. Consequently there exists an homomorphism h such that

$$\begin{aligned} h &: G \rightarrow \text{Aut}(\mathcal{X}) \\ h(g) &= s_g. \end{aligned} \quad (2.4)$$

Conversely, suppose that $h : G \rightarrow \text{Aut}(\mathcal{X})$ is a homomorphism. We define a map from $\mathcal{X} \times G \rightarrow \mathcal{X}$ by sending (x, g) to $h(g)(x)$. One can easily check that this map is an action of G on \mathcal{X} . It is an easy exercise now to show that the correspondence between the actions (2.2) and the homomorphisms (2.4) is bijective.

In the problem under consideration $G \equiv \mathcal{G}$ and $\mathcal{X} \equiv \mathcal{P} \simeq P_1(R) \times P_1(R)$, and, there is a natural bijective correspondence between the set of actions of \mathcal{G} on \mathcal{P} and the set of homomorphisms from \mathcal{G} into the group $\text{Aut}(\mathcal{P})$ of automorphisms of \mathcal{P} . Let h_m be the homomorphism which is associated with the specific action (2.1). The following Proposition gives the kernel of h_m .

Proposition 1 *The action from the right of \mathcal{G} on $P_1(R) \times P_1(R)$*

$$((x, y)(g, h)) \mapsto (xg, yh), \quad (2.5)$$

where $(g, h) \in \mathcal{G}$, $g \in G$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $xg = \frac{xa+c}{xb+d}$ is not effective. The kernel of h_m is

$$K = \{(g, h) \in \mathcal{G} \mid g = \pm I, \quad h = \pm I\}, \quad (2.6)$$

where I is the identity element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ of G .

Proof If $(g, h) \in K$, $xg = x$ and $yh = y$ for all (x, y) . So, if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $xg = \frac{xa+c}{xb+d} = x$ for all x . Taking $x = 0$ gives $\frac{c}{d} = 0$, and so, $c = 0$. Taking $x = \infty$ gives $\frac{a}{b} = \infty$, and therefore, $b = 0$. So we obtain that $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ (it is $\det g = 1$). Therefore we have $xg = \frac{ax+0}{0x+a^{-1}} = a^2x = x$. Setting $x = 1$ gives $a^2 = 1$, so $a = \pm 1$ and $g = \pm I$. Similarly $h = \pm I$. This completes the proof.

The group K is a normal subgroup of \mathcal{G} . The not effective action of \mathcal{G} on $\mathcal{P} \simeq P_1(R) \times P_1(R)$ passes naturally to an effective action of the group \mathcal{G}/K on the torus \mathcal{P} . This last action is given by

$$(x, y)((g, h)K) = (xg, yh), \quad (2.7)$$

where $(g, h)K$ denotes an element of the coset space \mathcal{G}/K . We will prove now that the action (2.7) not only is it effective but also fixed point free (f.p.f). To prove this we first need the following lemma.

Lemma 1 *When \mathcal{G} is restricted to its subgroup $SO(2) \times SO(2)$ the action (2.7) is f.p.f.*

Proof To simplify notation we denote $\left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \right)$ by $(R(\theta), R(\varphi))$. If $(g, h) = (R(\theta_o), R(\varphi_o))$ is an element of $SO(2) \times SO(2)$ then the action (2.1) reads

$$(\rho, \sigma) \mapsto (\rho + 2\theta_o, \sigma + 2\varphi_o),$$

where $x = \cot \frac{\rho}{2}$, $y = \cot \frac{\sigma}{2}$. Suppose that for some point (ρ_o, σ_o) of the torus $P_1(R) \times P_1(R)$ we have

$$(\rho_o + 2\theta_o, \sigma_o + 2\varphi_o) = (\rho_o, \sigma_o), \quad (2.8)$$

for some element $(R(\theta_o), R(\varphi_o))$ of $SO(2) \times SO(2)$. Since, ρ_o and σ_o are defined only mod 2π Eq. (2.8) can be satisfied for some point (ρ_o, σ_o) of the torus $P_1(R) \times P_1(R)$ if and only if $(R(\theta_o), R(\varphi_o))$ is an element of the kernel K of the homomorphism h_m . This completes the proof.

Now we are ready to prove the following

Lemma 2 *The action (2.7) is f.p.f.*

Proof Suppose that for a point (x_o, y_o) of $P_1(R) \times P_1(R)$ and an element (g_o, h_o) of \mathcal{G} we have

$$(x_o, y_o)(g_o, h_o) = (x_o g_o, y_o h_o) = (x_o, y_o). \quad (2.9)$$

The element $g_o = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, of $SL(2, R)$ can always be written in the form

$$g_o = u_o \delta_o w_o, \quad \delta_o = \delta_o(t) = \begin{bmatrix} e^{t_o/2} & 0 \\ 0 & e^{-t_o/2} \end{bmatrix}, \quad (2.10)$$

where $u_o, w_o \in SO(2)$ and t_o is real. We have

$$x_o \delta_o = \frac{x_o e^{t_o/2}}{e^{-t_o/2}} = x_o e^{t_o}. \quad (2.11)$$

When

$$x_o e^{t_o} = x_o$$

and $x_o \neq 0$ we obtain

$$e^{t_o} = 1 \Leftrightarrow t_o = 0.$$

From the last Equation and Lemma 1 we conclude that for almost all (x_o, y_o) when Eq. (2.9) is satisfied then

$$g_o = I \quad \text{or} \quad g_o = -I.$$

Similarly,

$$h_o = I \quad \text{or} \quad h_o = -I.$$

We conclude that for almost all (x_o, y_o) when Eq. (2.9) is satisfied then

$$(g_o, h_o) \in K,$$

where K is the kernel of the homomorphism h_m . This completes the proof.

It will prove convenient to recall at this point one definition and two propositions from elementary group theory (see e.g. [19] pages 4, 6 and 11). The symbol \leq denotes subgroup, the symbol \trianglelefteq denotes normal subgroup, whereas the symbol \cong denotes isomorphism.

Definition 1 *If X and Y are subgroups of a group G , then we define the product of X and Y in G to be $XY = \{xy \mid x \in X, y \in Y\} \subseteq G$.*

Proposition 2 *Let H and K be subgroups of a group G . If $K \trianglelefteq G$, then $HK \leq G$ and $H \cap K \trianglelefteq H$; if also $H \trianglelefteq G$, then $HK \trianglelefteq G$ and $H \cap K \trianglelefteq G$.*

Proposition 3 *Let G be a group. If $K \trianglelefteq G$ and $H \leq G$, then $HK/K \cong H/H \cap K$.*

Henceforth we will assume that the group G is finite. Let $|G|$ denote the order of a group G . We can prove now the following

Proposition 4 *Let G be a finite group. If $K \trianglelefteq G$ and $H \leq G$, then the number $|G|/|H|$ is divisible by the number $|K|/|K \cap H|$.*

Proof The number $|G|/|H|$ is divisible by the number $|K|/|K \cap H|$ if and only if the number $|G|/|K|$ is divisible by the number $|H|/|K \cap H|$. According to Proposition 2 the subset HK of G is a subgroup of G . The coset spaces G/K and HK/K can be equipped with a group structure since by assumption $K \trianglelefteq G$. Since $HK \leq G$, the group HK/K is a subgroup of G/K . Therefore the number $|G|/|H|$ is divisible by the number $|HK|/|K|$. But according to Proposition 3

$$|HK|/|K| = |H|/|K \cap H|.$$

This completes the proof.

There are $|H|^{\frac{|G|}{|H|}}$ different ways to choose coset representatives from the coset space G/H . Let \mathcal{S} be the set

$$\mathcal{S} = \left\{ S_1, S_2, \dots, S_{|H|^{\frac{|G|}{|H|}}} \right\}$$

whose elements are the different choices of coset representatives from the coset space G/H . For our needs we need to select specific elements of \mathcal{S} . We proceed now to give an explicit description of these elements.

Consider the coset space $K/K \cap H$. There are $|K \cap H|^{\frac{|K|}{|K \cap H|}}$ different ways to choose coset representatives from the coset space $K/K \cap H$. Let \mathfrak{S} be the set

$$\mathfrak{S} = \left\{ \sigma_1, \sigma_2, \dots, \sigma_{|K \cap H|^{\frac{|K|}{|K \cap H|}}} \right\}$$

whose elements are the different choices of coset representatives from the coset space $K/K \cap H$. Let us denote by

$$g_i \sigma_j$$

the elements of G which are obtained by multiplying the element g_i of G with the specific choice of coset representatives σ_j ($j \in \left\{1, 2, \dots, |\mathbf{K} \cap \mathbf{H}|^{\frac{|\mathbf{K}|}{|\mathbf{K} \cap \mathbf{H}|}}\right\}$). Therefore $g_i \sigma_j$ denotes collectively $\frac{|\mathbf{K}|}{|\mathbf{K} \cap \mathbf{H}|}$ elements of G . We write them explicitly as follows

$$g_i \sigma_{j1}, g_i \sigma_{j2}, \dots, g_i \sigma_{j\omega},$$

where $\omega \equiv \frac{|\mathbf{K}|}{|\mathbf{K} \cap \mathbf{H}|}$. The following 5 steps describe in algorithmic fashion the way we make our specific choices of elements of \mathcal{S} .

1. Pick up an element g_{i_1} of G and an element σ_{i_1} of \mathfrak{S} and construct the cosets

$$g_{i_1} \sigma_{i_1 1} \mathbf{H}, g_{i_1} \sigma_{i_1 2} \mathbf{H}, \dots, g_{i_1} \sigma_{i_1 \omega} \mathbf{H} \quad (2.12)$$

of the coset space G/\mathbf{H} .

2. Choose an element g_{i_2} of G which *does not* belong to the previous cosets (2.12) and an element σ_{i_2} of \mathfrak{S} . The element σ_{i_2} might be identical to σ_{i_1} or different from it. Construct the cosets

$$g_{i_2} \sigma_{i_2 1} \mathbf{H}, g_{i_2} \sigma_{i_2 2} \mathbf{H}, \dots, g_{i_2} \sigma_{i_2 \omega} \mathbf{H} \quad (2.13)$$

of the coset space G/\mathbf{H} .

3. Choose an element g_{i_3} of G which *does not* belong to the previous cosets (2.12) and (2.13) and an element σ_{i_3} of \mathfrak{S} . The element σ_{i_3} might be identical either to σ_{i_1} or to σ_{i_2} , or, it could be different from both of them. Construct the cosets

$$g_{i_3} \sigma_{i_3 1} \mathbf{H}, g_{i_3} \sigma_{i_3 2} \mathbf{H}, \dots, g_{i_3} \sigma_{i_3 \omega} \mathbf{H} \quad (2.14)$$

of the coset space G/\mathbf{H} .

4. Repeat the same procedure $\frac{\frac{|\mathbf{G}|}{|\mathbf{H}|}}{\frac{|\mathbf{K}|}{|\mathbf{K} \cap \mathbf{H}|}} = \frac{|\mathbf{G}| |\mathbf{H} \cap \mathbf{K}|}{|\mathbf{H}| |\mathbf{K}|}$ times and obtain finally the following set of cosets

$$g_{i_\theta} \sigma_{i_\theta 1} \mathbf{H}, g_{i_\theta} \sigma_{i_\theta 2} \mathbf{H}, \dots, g_{i_\theta} \sigma_{i_\theta \omega} \mathbf{H} \quad (2.15)$$

of the coset space G/\mathbf{H} , where $\theta \equiv \frac{\frac{|\mathbf{G}|}{|\mathbf{H}|}}{\frac{|\mathbf{K}|}{|\mathbf{K} \cap \mathbf{H}|}} = \frac{|\mathbf{G}| |\mathbf{H} \cap \mathbf{K}|}{|\mathbf{H}| |\mathbf{K}|}$.

5. Consider the following set S_i of elements of G

$$S_i = \{g_{i_1} \sigma_{i_1 1}, g_{i_1} \sigma_{i_1 2}, \dots, g_{i_1} \sigma_{i_1 \omega}, g_{i_2} \sigma_{i_2 1}, g_{i_2} \sigma_{i_2 2}, \dots, g_{i_2} \sigma_{i_2 \omega}, \\ g_{i_3} \sigma_{i_3 1}, g_{i_3} \sigma_{i_3 2}, \dots, g_{i_3} \sigma_{i_3 \omega}, \dots, g_{i_\theta} \sigma_{i_\theta 1}, g_{i_\theta} \sigma_{i_\theta 2}, \dots, g_{i_\theta} \sigma_{i_\theta \omega}\}. \quad (2.16)$$

This completes the construction.

Let $S_{i'}$ be a set of elements of G , which if differs from S_i , it differs in the choice of the elements σ_ξ , $\xi \in \{1, 2, \dots, \theta\}$, chosen from the set \mathfrak{S} . We will write

$$S_i \sim S_{i'}. \quad (2.17)$$

Let \mathcal{C}_1 and \mathcal{C}_2 be the following sets of cosets

$$\mathcal{C}_1 = \{g_i \sigma_{i_1 1} H, g_i \sigma_{i_1 2} H, \dots, g_i \sigma_{i_1 \omega} H\} \quad (2.18)$$

and

$$\mathcal{C}_2 = \{g_i \sigma_{i_2 1} H, g_i \sigma_{i_2 2} H, \dots, g_i \sigma_{i_2 \omega} H\}, \quad (2.19)$$

where σ_1 and σ_2 are *distinct* elements of \mathfrak{S} . Then we have the following.

Proposition 5 *The following are true*

1. *Every element of \mathcal{C}_1 belongs to \mathcal{C}_2 (and vice versa).*
2. *The set S_i is an element of \mathcal{S} , i.e., the elements of S_i are coset representatives of the coset space G/H .*
3. *The relation (2.17) is an equivalence relation.*

Proof 1. Choose an element $g_i \sigma_{i_1 j} H$ of \mathcal{C}_1 (so $j \in \{1, 2, \dots, \omega\}$) This element belongs to \mathcal{C}_2 if and only if for given $\sigma_{i_1 j}$ there always exists an element $g_i \sigma_{i_2 j'} H$ of \mathcal{C}_2 ($j' \in \{1, 2, \dots, \omega\}$) which satisfies

$$g_i \sigma_{i_1 j} = g_i \sigma_{i_2 j'} h \Leftrightarrow \sigma_{i_1 j} = \sigma_{i_2 j'} h \quad (2.20)$$

for some $\sigma_{i_2 j'}$ and some $h \in H$. Now the element $\sigma_{i_1 j}$ of K *always* belongs to some (a unique) coset of the coset space $K/K \cap H$. Therefore, always there exists an element $\sigma_{i_2 j'}$ of σ_{i_2} such that $\sigma_{i_1 j} \in \sigma_{i_2 j'}(K \cap H)$. Consequently there always exist (unique) elements $\sigma_{i_2 j'}$ of σ_{i_2} and $h \in K \cap H$ which satisfy Eq. (2.20).

2. Consider the cosets

$$g_{i_\nu} \sigma_{i_\nu 1} H, g_{i_\nu} \sigma_{i_\nu 2} H, \dots, g_{i_\nu} \sigma_{i_\nu \omega} H, \quad (2.21)$$

where, $\nu \in \{1, 2, \dots, \theta\}$. Any two $g_{i_\nu}\sigma_{i_\nu\mu_1}H$, $g_{i_\nu}\sigma_{i_\nu\mu_2}H$, $\mu_1, \mu_2 \in \{1, 2, \dots, \omega\}$ of them are distinct. Indeed, suppose they are not. Then they coincide, and for every $h_1 \in H$, there always exists a unique $h_2 \in H$, such that

$$g_{i_\nu}\sigma_{i_\nu\mu_1}h_1 = g_{i_\nu}\sigma_{i_\nu\mu_2}h_2 \Leftrightarrow \sigma_{i_\nu\mu_2}^{-1}\sigma_{i_\nu\mu_1} = h_2h_1^{-1}. \quad (2.22)$$

Since $\sigma_{i_\nu\mu_2}^{-1}\sigma_{i_\nu\mu_1} = \sigma \in K$ we have that $h_2h_1^{-1} = \sigma \in K \cap H$. From Eq. (2.22) we obtain

$$\sigma_{i_\nu\mu_1} = \sigma_{i_\nu\mu_2}\sigma, \quad \sigma \in K \cap H.$$

Therefore, $\sigma_{i_\nu\mu_1}, \sigma_{i_\nu\mu_2}$ belong to the same coset of the coset space $K/K \cap H$. Contradiction. We conclude that any two $g_{i_\nu}\sigma_{i_\nu\mu_1}H$, $g_{i_\nu}\sigma_{i_\nu\mu_2}H$, $\mu_1, \mu_2 \in \{1, 2, \dots, \omega\}$, of the cosets (2.21) are different from one another.

Choose now a coset $g_{i_\tau}\sigma_{i_\tau j}H$ and without loss of generality assume that $\tau > \nu$. The coset $g_{i_\tau}\sigma_{i_\tau j}H$ is different from the cosets (2.21). Indeed, suppose that it coincides with one of them, say $g_{i_\nu}\sigma_{i_\nu k}H$ (where $k \in \{1, 2, \dots, \omega\}$). Then for every h_1 there always exists a unique $h_2 \in H$ such that

$$g_{i_\tau}\sigma_{i_\tau j}h_1 = g_{i_\nu}\sigma_{i_\nu k}h_2 \Leftrightarrow g_{i_\tau}\sigma_{i_\tau j} = g_{i_\nu}\sigma_{i_\nu k}h_2h_1^{-1} \quad (2.23)$$

By writing $h_2h_1^{-1} = h \in H$ the last equation gives

$$g_{i_\tau} = g_{i_\nu}\sigma_{i_\nu k}(h\sigma_{i_\tau j}^{-1}h^{-1})h. \quad (2.24)$$

Since, $\sigma_{i_\tau j}^{-1} \in K$, and since, $K \trianglelefteq G$ we have that $h\sigma_{i_\tau j}^{-1}h^{-1} \in K$, so say, $h\sigma_{i_\tau j}^{-1}h^{-1} = \sigma' \in K$. By writing $\sigma_{i_\nu k}\sigma' = \bar{\sigma}$ Eq. (2.24) gives

$$g_{i_\tau} = g_{i_\nu}\bar{\sigma}h$$

for some $h \in H$. This implies that $g_{i_\tau} \in g_{i_\nu}\bar{\sigma}H$. In 1 it was shown that the coset $g_{i_\nu}\bar{\sigma}H$ coincides with one of the cosets (2.21). Therefore, g_{i_τ} belongs to one of the cosets (2.21). Contradiction, since by assumption $\tau > \nu$, and therefore, the element g_{i_τ} does not belong to any of the cosets (2.21).

3. Let S_1 , S_2 and S_3 be three sets of elements of G of the form (2.16). Assume that $S_1 \sim S_2$ and also that $S_2 \sim S_3$. The relation defined in Eq. (2.17) is reflexive ($S_1 \sim S_1$), symmetric ($S_1 \sim S_2$ implies $S_2 \sim S_1$), and transitive ($S_1 \sim S_2$ and $S_2 \sim S_3$ together, imply $S_1 \sim S_3$). This completes the proof.

Let $\overline{\mathcal{S}}$ be the subset of \mathcal{S} which has only members of the form (2.16). The relation defined in Eq. (2.17) is an equivalence relation on $\overline{\mathcal{S}}$. If $S \in \overline{\mathcal{S}}$, then \tilde{S} denotes the equivalence class of S .

In our study the finite group G is the group $C_n \times C_m$ which acts on the torus $\mathcal{P} \simeq P_1(R) \times P_1(R)$. With a view to apply the previous theory to the problem under consideration we leave the group G to act on any manifold (the theorems proven here are slightly more general than it is strictly needed, in fact we allow G to act on any topological space) and we establish a few more facts. The details are as follows.

Let M be any topological space, and let G be any finite group which acts on M from the right. That is to say, we are given a map $M \times G \rightarrow M$, denoted $(x, g) \mapsto xg$, with the following properties. For each $g \in G$, the map $x \mapsto xg$ is a homeomorphism of M onto itself. If $g = e$ (the identity element), $xe = x$ for every $x \in M$. For any $x \in M$ and $g_1, g_2 \in G$, $(xg_1)g_2 = x(g_1g_2)$. There is an homomorphism h_G from G into the group of automorphisms $\text{Aut}(M)$ of M which is naturally associated with this action. Let K be the kernel of h_G . The action of G on M passes naturally to an action of G/K on M . Henceforth, we will assume that this action

$$x(gK) = xg,$$

where, $x \in M$, and $g \in G$ is fixed point free. An *elementary domain* for the given action is an open subset $E \subset M$ such that the following conditions are satisfied:

$$\begin{aligned} \text{(A)} & \text{ For any } g_1, g_2 \in G, \text{ with } g_1 \neq g_2k, \ k \in K, \ E g_1 \cap \overline{E} g_2 = \emptyset, \\ \text{(B)} & \bigcup_{g \in G} \overline{E} g = M. \end{aligned} \tag{2.25}$$

Here bar means topological closure, and \emptyset means the empty set. Now let $H \subset G$ be any subgroup of G , and let $S \subset G$ be a set of representatives of the left cosets of H which is a member of $\overline{\mathcal{S}}$. Since $S \subset G$ is a set of representatives of the left cosets of H in the coset space G/H we have

$$\begin{aligned} \text{(C)} & G = SH, \\ \text{(D)} & SH = \bigcup_{s \in S} sH \quad (\text{disjoint union}). \end{aligned} \tag{2.26}$$

Then we have the following result relating elementary domains for G and H :

Proposition 6 *ES is an elementary domain for H .*

Proof First note that condition (B) can be written $\overline{E}G = M$. To verify condition (B) for H , we calculate, using (C), as follows. $(\overline{E}S)H = \overline{E}(SH) = \overline{E}G = M$, as required. To verify condition (A) for H , we must prove that, for any $h_1, h_2 \in H$ with $h_1 \neq h_2k_H$, $k_H \in K \cap H$, $(ES)h_1$ and $(\overline{E}S)h_2 = (\overline{E}S)h_2$ are disjoint. Assume that they are not disjoint. Then there exists an element $x \in (ES)h_1 \cap (\overline{E}S)h_2$. So, by definition, $x = z_1s_1h_1 = z_2s_2h_2$ for some $z_1 \in E$, $z_2 \in \overline{E}$ and $s_1, s_2 \in S$. Write $g_1 = s_1h_1$ and $g_2 = s_2h_2$. Then $z_1g_1 = z_2g_2$, so $Eg_1 \cap \overline{E}g_2 \neq \emptyset$. So, by property (A), $g_1 = g_2k$, $k \in K$. That is,

$$s_1h_1 = s_2h_2k. \quad (2.27)$$

We distinguish two cases.

1. Assume that $H \cap K = K$. Then Eq. (2.27) gives

$$s_1H = s_2H \implies s_1 = s_2.$$

Substituting back into Eq. (2.27) we obtain $h_1 = h_2k$, where $k \in H \cap K = K$. This contradicts $h_1 \neq h_2k$, and so $(ES)h_1$ and $(\overline{E}S)h_2$ are actually disjoint; condition (A) is satisfied for H . Henceforth we can assume that $H \cap K < K$, (the symbol $<$ indicates proper subgroup), and that the coset space $K/H \cap K$ has at least two elements.

2. Assume that $H \cap K < K$. Eq. (2.27) implies that

$$s_1h_1k^{-1} = s_2h_2. \quad (2.28)$$

Now $k^{-1} \in K$ and K is a normal subgroup of G . Therefore, $h_1k^{-1} = k'h_1$, for some $k' \in K$. Substituting into Eq. (2.28) we obtain

$$s_1k'h_1 = s_2h_2. \quad (2.29)$$

We distinguish three cases.

- 2a. Assume now that $s_1k' \in S$, where S is the fixed set of coset representatives we chose at the beginning. Then Eq. (2.29) gives

$$s_1k'H = s_2H \implies s_1k' = s_2. \quad (2.30)$$

Substituting into Eq. (2.29) we obtain $h_1 = h_2$. This contradicts $h_1 \neq h_2k$, and so henceforth we can assume that $s_1k' \notin S$.

2b. Assume now that $s_1k' \notin S$ and also that $s_1k' \in s_1H$. Then there exists $h \in H$ such that $s_1k' = s_1h$. Therefore, $k' = h$ and $k' \in K \cap H$. According to Proposition 2 the group $K \cap H$ is normal in H , and therefore, $k'h_1 = h_1\bar{k}$, for some \bar{k} in $K \cap H$. Substituting back into Eq. (2.29) we obtain

$$s_1h_1 = s_2h_2\bar{k}^{-1}. \quad (2.31)$$

From the last Equation we obtain

$$s_1H = s_2H \implies s_1 = s_2.$$

Substituting back into Eq. (2.31) we obtain

$$h_1 = h_2\bar{k}^{-1}.$$

This contradicts $h_1 \neq h_2k$, and so henceforth we can assume that $s_1k' \notin S$, and that $s_1k' \notin s_1H$.

2c. Assume now that $s_1k' \notin S$ and also that $s_1k' \notin s_1H$. From Eq. (2.29) we obtain

$$s_1k' \in s_2H. \quad (2.32)$$

From Proposition 5 (statement 1) we have that

$$s_1k' \in s_1k_oH, \quad (2.33)$$

for some (unique) $s_1k_o \in S$. From Equations (2.32) and (2.33) we have $s_2H = s_1k_oH$ and therefore we obtain $s_2 = s_1k_o$. Substituting back into Eq. (2.29) we obtain

$$k'h_1 = k_oh_2.$$

The last Equation gives $k_o^{-1}k' = h_2h_1^{-1}$. Therefore we have $k_o^{-1}k' = k'' \in K \cap H$. So we have $k''h_1 = h_2$. The group $K \cap H$ is normal in H and therefore $k''h_1 = h_1\bar{\bar{k}}$, for some $\bar{\bar{k}} \in K \cap H$. So finally we obtain

$$h_1 = h_2\bar{\bar{k}}^{-1}.$$

This contradicts $h_1 \neq h_2k$, $k \in K \cap H$. So $(ES)h_1$ and $(\bar{E}S)h_2$ are actually disjoint; condition (A) is satisfied for H . This completes the proof.

3 Invariant Subspaces for Finite Groups

For reasons which will become clear later, in this section we take the positive integers n and m to be even. Consider the particular (non-effective) action $S^1 \times C_n \rightarrow S^1$ given by

$$(\rho, g_r) \mapsto \rho g_r = \rho + \frac{4\pi}{n}r, \quad (3.1)$$

where ρ is the usual angular coordinate on the circle, taken mod 2π , and $g_r \in C_n$ are the elements of C_n , where $0 \leq r \leq (n-1)$. Let h' be the homomorphism associated with the action (3.1). One can easily show that the kernel K_{C_n} of this homomorphism is $K_{C_n} = \{I, -I\}$, where I is the identity element of the group C_n . Now we show that

Proposition 7 *The open set*

$$E_n = \{\rho \in S^1 \mid 0 < \rho < 4\pi/n\} \quad (3.2)$$

is elementary for the action (3.1).

Proof Let g_1 and g_2 be two elements of C_n . Then we have

$$\begin{aligned} E_n g_1 &= \left\{ \rho \in S^1 \mid \rho = \rho' + \frac{4\pi}{n}r_1, \rho' \in E_n \right\} \\ &= \left\{ \rho \in S^1 \mid \frac{4\pi}{n}r_1 < \rho < \frac{4\pi}{n} + \frac{4\pi}{n}r_1 \right\} \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \bar{E}_n g_2 &= \left\{ \rho \in S^1 \mid \rho = \rho' + \frac{4\pi}{n}r_2, \rho' \in \bar{E}_n \right\} \\ &= \left\{ \rho \in S^1 \mid \frac{4\pi}{n}r_2 \leq \rho \leq \frac{4\pi}{n} + \frac{4\pi}{n}r_2 \right\}. \end{aligned} \quad (3.4)$$

Assume now that $E_n g_1 \cap \bar{E}_n g_2 \neq \emptyset$. Then from Eqs. (3.3) and (3.4) we conclude that there must exist some $\rho_o \in E_n$ such that

$$\rho_o + \frac{4\pi}{n}r_1 = \rho_o + \frac{4\pi}{n}r_2 + a2\pi, \quad (3.5)$$

for some integer a , since ρ is only defined mod 2π . Eq. (3.5) gives

$$r_1 - r_2 = a \frac{n}{2}. \quad (3.6)$$

The last Equation is equivalent to

$$g_1 = g_2 k, \quad (3.7)$$

where $k \in K_{C_n}$, and K_{C_n} is the kernel of the homomorphism h' . Therefore the condition (A) of Eq. (2.25) is satisfied. We note that that *every* $\rho_o \in S^1$ belongs to some interval of the form

$$\bar{E}_n g_r = \left\{ \rho \in S^1 \mid \frac{4\pi}{n} r \leq \rho \leq \frac{4\pi}{n} + \frac{4\pi}{n} r \right\},$$

where, $r \in \{0, 1, 2, \dots, \frac{n}{2} - 1\}$. We conclude that

$$\bigcup_{g_r \in C_n} \bar{E}_n g_r = S^1. \quad (3.8)$$

(In fact Eq. (3.8) is satisfied even when r runs only through the set of values $\{0, 1, 2, \dots, \frac{n}{2} - 1\}$). Therefore the condition (B) of Eq. (2.25) is satisfied. This completes the proof.

Next, consider the (non-effective) action $\mathcal{P} \times (C_n \times C_m) \rightarrow \mathcal{P}$ given by

$$((\rho, \sigma), (g_i, g_j)) \mapsto (\rho g_i, \sigma g_j) \quad (3.9)$$

where $0 \leq i \leq (n-1)$, $0 \leq j \leq (m-1)$, $g_i \in C_n$, $g_j \in C_m$ and

$$\rho g_i = \rho + \frac{4\pi}{n} i, \quad \sigma g_j = \sigma + \frac{4\pi}{m} j. \quad (3.10)$$

Define the set $F_{nm} \subset \mathcal{P}$ by the formula

$$F_{nm} = E_n \times E_m \subset \mathcal{P}. \quad (3.11)$$

We prove now that

Proposition 8 F_{nm} is an elementary domain for the action (3.9).

Proof Since E_n and E_m are open, so is F_{nm} . If $(g_i, g_j) \neq (g_{i'}, g_{j'})k$, where $k \in K = \{(I, I), (-I, -I), (I, -I), (-I, I)\}$, then either $g_i \neq g_{i'}\lambda$ or $g_j \neq g_{j'}\lambda$ or both, where $\lambda \in K_{C_n} = \{I, -I\}$. We have

$$F_{nm}(g_i, g_j) = (E_n g_i) \times (E_m g_j), \quad (3.12)$$

$$\bar{F}_{nm}(g_{i'}, g_{j'}) = (\bar{E}_n g_{i'}) \times (\bar{E}_m g_{j'}). \quad (3.13)$$

Taking the intersection gives

$$(E_n g_i \times E_m g_j) \cap (\overline{E}_n g_{i'} \times \overline{E}_m g_{j'}) = (E_n g_i \cap \overline{E}_n g_{i'}) \times (E_m g_j \cap \overline{E}_m g_{j'}). \quad (3.14)$$

Since E_n is elementary for the original action of C_n on S^1 , at least one of the factors on the RHS is empty. So condition (A) is satisfied. For (B), let $(\rho, \sigma) \in T^2$ be given. Then, because E_n is elementary for the original action, $\rho \in \overline{E}_n g_i$ for some $i = 0, 1, 2, \dots, (n-1)$, and $\sigma \in \overline{E}_m g_j$ for some $j = 0, 1, 2, \dots, (m-1)$. So

$$(\rho, \sigma) \in \overline{E}_n g_i \times \overline{E}_m g_j = \overline{F}_{nm}(g_i, g_j). \quad (3.15)$$

That is, every (ρ, σ) belongs to some $\overline{F}_{nm}(g_i, g_j)$ and so condition (B) is satisfied. This completes the proof.

Now let $F \subset C_n \times C_m$ be any subgroup of $C_n \times C_m$. Let S be a selection of representatives of left cosets of F which belongs to \overline{S} . Then, by Proposition 6, the set

$$E = F_{nm} S \subset \mathcal{P} \quad (3.16)$$

is an elementary domain for the subgroup F . Let $\mathcal{H}(\mathcal{P})$ denote the Hilbert space of functions $f : \mathcal{P} \rightarrow R$ which are square integrable with respect to the usual Lesbegue measure $d\theta \wedge d\phi$ on the torus $\mathcal{P} \simeq P_1(R) \times P_1(R)$. We now want to find the Hilbert space $\mathcal{H}(F)$ of all invariant vectors for each of the finite subgroups F of $C_n \times C_m$. Thus, $\mathcal{H}(F)$ is the space

$$\mathcal{H}(F) = \left\{ \tilde{\zeta} \in \mathcal{H}(\mathcal{P}) \mid T'(h)\tilde{\zeta} = \tilde{\zeta} \text{ for all } h \in F \right\}. \quad (3.17)$$

$\mathcal{H}(F)$ is a closed subspace of $\mathcal{H}(\mathcal{P})$. First note that $E = F_{nm} S$ is a union of open rectangles in \mathcal{P} , so inherits the Lesbegue measure $d\theta \wedge d\phi$ from \mathcal{P} . Let $\mathcal{H}(E)$ denote the Hilbert space of square integrable functions $f : E \rightarrow R$, and $\mathcal{H}_E(\mathcal{P})$ the Hilbert subspace of $\mathcal{H}(\mathcal{P})$ consisting of functions which vanish outside E ;

$$\mathcal{H}_E(\mathcal{P}) = \{f \in \mathcal{H}(\mathcal{P}) \mid f(x) = 0, \text{ all } x \notin E\}. \quad (3.18)$$

There is a bijection between $\mathcal{H}(E)$ and $\mathcal{H}_E(\mathcal{P})$, obtained as follows. Define maps $\alpha : \mathcal{H}(E) \rightarrow \mathcal{H}_E(\mathcal{P})$ and $\beta : \mathcal{H}_E(\mathcal{P}) \rightarrow \mathcal{H}(E)$ by “extending to 0 outside E ” and “restricting to E ” respectively:

$$(\alpha(l))(x) = \begin{cases} l(x), & x \in E \\ 0, & x \notin E \end{cases}, \quad (3.19)$$

$$(\beta(f))(x) = f(x), \quad x \in E. \quad (3.20)$$

Here typical elements are denoted by $l \in \mathcal{H}(E)$ and $f \in \mathcal{H}_E(\mathcal{P})$. Then routine checks show that $(\beta\alpha)(l) = l$ for all l , and $(\alpha\beta)(f) = f$ for all f ; $\beta\alpha$ is the identity map on $\mathcal{H}(E)$, $\alpha\beta$ is the identity map on $\mathcal{H}_E(\mathcal{P})$. Thus there is an induced bijection

$$\mathcal{H}(E) \leftrightarrow \mathcal{H}_E(\mathcal{P}). \quad (3.21)$$

Before our next result, note first that, from the conditions (A) and (B) defining an elementary region, the union

$$M = \bigcup_{gK \in G/K} \bar{E}(gK) \quad (3.22)$$

is “almost disjoint”, in the sense that overlaps between the sets $\bar{E}g_1$ and $\bar{E}g_2$, for any $g_1 \neq g_2k$, $k \in K$, can only involve boundary points of E . In our situation ($M = \mathcal{P}$, $F \subset G = C_n \times C_m$ and $K = \{(I, I), (I, -I), (-I, I), (-I, -I)\}$), these sets (involving boundary points) are, at most, one dimensional, and so certainly of measure zero.

The representation theory of \mathcal{HB} is governed by the dual action T' of $\mathcal{G} = \text{SL}(2, R) \times \text{SL}(2, R)$ on $L^2(\mathcal{P}, \lambda, R) \simeq L^2(\mathcal{P}, \lambda, R)$ given by (Eq. (1.10))

$$(T'(g, h)\phi)(x, y) = k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y)\phi(xg, yh).$$

Subsequently, for notational simplicity, we will write $(T'(g)f)(x) = \gamma(x, g)f(xg)$, and it will be understood that by $\gamma(x, g)$ we denote the multiplier

$$\gamma(x, g) = k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y), \quad (3.23)$$

and that when we write $(T'(g)f)(x) = \gamma(x, g)f(xg)$, by x we denote a point of the torus \mathcal{P} , and by g an element of \mathcal{G} .

Now we define two maps ρ and σ (we use the same symbols to denote the usual angular coordinates on the torus \mathcal{P} ; hopefully the context will cause no doubts regarding their meaning when they are encountered) as follows

$$\begin{aligned} \sigma & : \mathcal{H}_E(\mathcal{P}) \rightarrow \mathcal{H}(F) \\ \sigma(f) & = \frac{1}{|K_F|} \sum_{g \in F} T'(g)f, \end{aligned} \quad (3.24)$$

where f is an element of $\mathcal{H}_E(\mathcal{P})$, and $|K_F|$ denotes the order of the group $K_F = F \cap K$. The map ρ is defined as follows

$$\begin{aligned} \rho & : \mathcal{H}(F) \rightarrow \mathcal{H}_E(\mathcal{P}) \\ (\rho(f))(x) & = \chi_E(x)f(x) = (\chi_E \cdot f)(x), \end{aligned} \quad (3.25)$$

where $x \in \mathcal{P}$, $f \in \mathcal{H}(F)$, and χ_E denotes the characteristic function of E , equal to 1 inside E , and 0 outside. The dot denotes pointwise multiplication. We now prove the following theorem

Theorem 2 *Let $G = C_n \times C_m$ and let F be a subgroup of G . Let $h_G : G \mapsto \text{Aut}(\mathcal{P})$, $\text{Aut}(\mathcal{P})$ is the group of automorphisms of \mathcal{P} , be the homomorphism which is associated with the action (3.9). Let K be the kernel of the homomorphism h_G . Let K_F denote the intersection $K \cap F$. Then we have the following*

1. The map $\rho\sigma$, computed on $\mathcal{H}_E(\mathcal{P})$, gives

$$(\rho(\sigma(f)))(x) = \frac{1}{|K_F|} [\gamma(x, e) + \gamma(x, k_2) + \gamma(x, k_3) + \dots + \gamma(x, k_{|K_F|})] f(x), \quad (3.26)$$

where, $x \in \mathcal{P}$, $f \in \mathcal{H}_E(\mathcal{P})$, $|K_F|$ denotes the order of the group $K \cap F$ and $K \cap F = \{e, k_2, k_3, \dots, k_{|K_F|}\}$.

2. The map $\sigma\rho$, computed on $\mathcal{H}(F)$, gives

$$(\sigma(\rho(f)))(x) = f(x), \quad (3.27)$$

where, $x \in \mathcal{P}$ and $f \in \mathcal{H}(F)$.

Proof Note that σ is well defined since, for any $g_0 \in F$,

$$\begin{aligned} T'(g_0)[\sigma(f)] &= T'(g_0) \left\{ \frac{1}{|K_F|} \sum_{g \in F} T'(g)f \right\} = \frac{1}{|K_F|} \sum_{g \in F} T'(g_0)T'(g)f = \\ &= \frac{1}{|K_F|} \sum_{g \in F} T'(g_0g)f. \end{aligned} \quad (3.28)$$

But, as g runs over F , so does g_0g , so the final sum is just $\sigma(f)$. So $\sigma(f) \in \mathcal{H}(F)$, as required. We first compute $\rho\sigma$. We have

$$\begin{aligned} (\sigma(f))(x) &= \frac{1}{|K_F|} \sum_{g \in F} (T'(g)f)(x) = \frac{1}{|K_F|} \sum_{g \in F} \gamma(x, g)f(xg) \\ &= \frac{1}{|K_F|} \{ \gamma(x, e)f(xe) + \gamma(x, g_2)f(xg_2) + \gamma(x, g_3)f(xg_3) + \\ &\quad + \dots + \gamma(x, g_n)f(xg_n) \}, \end{aligned} \quad (3.29)$$

where $e = g_1, g_2, g_3, \dots, g_n$ are the elements of F . Also $\rho(\sigma(f)) = \chi_E(\sigma(f))$, so, evaluating at x ,

$$\begin{aligned} (\rho(\sigma(f)))(x) &= \chi_E(x)(\sigma(f))(x) = \\ &= \frac{1}{|K_F|} \chi_E(x) [\gamma(x, e)f(xe) + \gamma(x, g_2)f(xg_2) + \gamma(x, g_3)f(xg_3) + \\ &+ \dots + \gamma(x, g_n)f(xg_n)]. \end{aligned} \quad (3.30)$$

If $x \in E$, $xg_i \notin E$ when $g_i \notin K_F$, $i = 2, 3, \dots, n$, because E is elementary. Therefore, the RHS of Eq. (3.30) equals to

$$\begin{aligned} (\rho(\sigma(f)))(x) &= \chi_E(x)(\sigma(f))(x) = \\ &= \frac{1}{|K_F|} \chi_E(x) [\gamma(x, e)f(xe) + \gamma(x, k_2)f(xk_2) + \gamma(x, k_3)f(xk_3) + \\ &+ \dots + \gamma(x, k_{|K_F|})f(xk_{|K_F|})], \end{aligned} \quad (3.31)$$

where $K \cap F = \{e, k_2, k_3, \dots, k_{|K_F|}\}$. If $x \notin E$, the RHS of Eq. (3.30) is 0. Since $f \in \mathcal{H}_E(\mathcal{P})$, this means that for all such f ,

$$(\rho(\sigma(f)))(x) = \frac{1}{|K_F|} [\gamma(x, e) + \gamma(x, k_2) + \gamma(x, k_3) + \dots + \gamma(x, k_{|K_F|})] f(x), \quad (3.32)$$

where, $x \in \mathcal{P}$, $f \in \mathcal{H}_E(\mathcal{P})$, $|K_F|$ denotes the order of the group $K \cap F$ and $K \cap F = \{e, k_2, k_3, \dots, k_{|K_F|}\}$, as required.

Next, we compute $\sigma\rho$. We have

$$\sigma(\rho(f)) = \frac{1}{|K_F|} \sum_{g \in F} T'(g)(\rho(f)) = \frac{1}{|K_F|} \sum_{g \in F} T'(g)(\chi_E \cdot f). \quad (3.33)$$

Evaluating at x ,

$$\begin{aligned} (\sigma(\rho(f)))(x) &= \frac{1}{|K_F|} \sum_{g \in F} (T'(g)(\chi_E \cdot f))(x) = \frac{1}{|K_F|} \sum_{g \in F} \gamma(x, g)(\chi_E \cdot f)(xg) = \\ &= \frac{1}{|K_F|} \sum_{g \in F} \gamma(x, g) \chi_E(xg) f(xg). \end{aligned} \quad (3.34)$$

But f is invariant, so $\gamma(x, g)f(xg) = f(x)$ for all $x \in \mathcal{P}$, so the sum on the RHS becomes

$$\frac{1}{|K_F|} f(x) \sum_{g \in F} \chi_E(xg) = \frac{1}{|K_F|} f(x) [\chi_E(x) + \chi_E(xg_2) + \dots + \chi_E(xg_n)]. \quad (3.35)$$

By (2.25), almost every $x \in \mathcal{P}$ belongs to $|K_F|$ open sets of the form Eg_i^{-1} ($i = 1, 2, \dots, n$ and $n = |F|$). Thus $xg_i \in E$ for $|K_F|$ values of i , for almost all x . The sum

in square brackets thus, almost always, contributes $|K_F|$, (1 for each $\chi_E(xg_i)$, i being one of these $|K_F|$ values, and 0 for all the other terms). So, for almost all x , the sum is $|K_F|$, and so the RHS is $f(x)$. Hence

$$(\sigma(\rho(f)))(x) = f(x) \quad (3.36)$$

for almost all x , and so, as Hilbert space maps, $\sigma\rho(f) = f$ for all $f \in \mathcal{H}(F)$. This completes the proof.

4 Finite Potential Little Groups

Define the groups A , B , C , and D , as follows $A = \{(I, I), (-I, -I)\}$, $B = \{(I, I), (-I, I)\}$, $C = \{(I, I), (I, -I)\}$, and, $D = \{(I, I), (-I, I), (I, -I), (-I, -I)\}$. The subgroups of $C_n \times C_m$ fall into one of the following disjoint classes:

1. **Class O.** This class contains those subgroups of $C_n \times C_m$ which are of odd order.
2. **Class E₁.** This class contains those subgroups of $C_n \times C_m$ which are of even order and contain the group A but they do not contain the group D .
3. **Class E₂.** This class contains those subgroups of $C_n \times C_m$ which are of even order and contain the group B but they do not contain the group D .
4. **Class E₃.** This class contains those subgroups of $C_n \times C_m$ which are of even order and contain the group C but they do not contain the group D .
5. **Class E₄.** This class contains those subgroups of $C_n \times C_m$ which are of even order and contain the group D .

From Eq. (1.10) we have that

$$(T'(-I, -I)\phi)(x, y) = \phi(x, y). \quad (4.1)$$

We also have

$$(T'(-I, I)\phi)(x, y) = -\phi(x, y), \quad (4.2)$$

and

$$(T'(I, -I)\phi)(x, y) = -\phi(x, y). \quad (4.3)$$

Eq. (4.1) implies that every potential little group contains the element $(-I, -I)$. Eq. (4.2) implies that the invariant subspace of the groups which belong to the class **Class E₂** consists just of one element, namely, the zero function. Eq. (4.3) implies that the invariant subspace of the groups which belong to the class **Class E₃** consists just of one element, namely, the zero function. Finally, both Eq. (4.2) and Eq. (4.3) imply that the invariant subspace of the groups which belong to the class **Class E₄** consists just of one element, namely, the zero function. Now we examine in turn the invariant subspaces of the groups which belong to the **Class E₁** and to the class **Class O**.

Class E₁. If F is a group which belongs to this class then $K_F \equiv A = \{(I, I), (-I, -I)\}$, and $|K_F| = 2$. Moreover, from Eq. (3.23) we have that

$$\gamma(x, e) = \gamma(x, k_2) = 1, \quad (4.4)$$

where $k_2 = (-I, -I)$. Substituting the result of Eq. (4.4) into Eq. (3.32) and taking into account that $|K_F| = 2$ we obtain

$$(\rho(\sigma(f)))(x) = f(x), \quad (4.5)$$

where, $x \in \mathcal{P}$, and $f \in \mathcal{H}_E(\mathcal{P})$. Moreover, Eq. (3.36) reads

$$(\sigma(\rho(f)))(x) = f(x)$$

for all $f \in \mathcal{H}(F)$. From equations (4.5) and (3.36) we conclude that there are two maps, namely, $\sigma : \mathcal{H}_E(\mathcal{P}) \rightarrow \mathcal{H}(F)$ and $\rho : \mathcal{H}(F) \rightarrow \mathcal{H}_E(\mathcal{P})$ such that $\rho\sigma$ is the identity map on $\mathcal{H}_E(\mathcal{P})$, and $\sigma\rho$ is the identity map on $\mathcal{H}(F)$. Therefore, we have the bijection

$$\mathcal{H}_E(\mathcal{P}) \leftrightarrow \mathcal{H}(F).$$

Combining this bijection with (3.21), we have the following bijections:

$$\mathcal{H}(E) \leftrightarrow \mathcal{H}_E(\mathcal{P}) \leftrightarrow \mathcal{H}(F), \quad (4.6)$$

which completely describes all invariant functions for F ; they are just given by “arbitrary” functions on the elementary region E for F .

Class O. If F is a group which belongs to this class then $K_F = \{(I, I)\}$, and $|K_F| = 1$. Moreover, from Eq. (3.23) we have that

$$\gamma(x, e) = 1, \quad (4.7)$$

where $e = (I, I)$. Substituting the result of Eq. (4.7) into Eq. (3.32) and taking into account that $|K_F| = 1$ we obtain

$$(\rho(\sigma(f)))(x) = f(x), \quad (4.8)$$

where, $x \in \mathcal{P}$, and $f \in \mathcal{H}_E(\mathcal{P})$. Moreover, Eq. (3.36) reads

$$(\sigma(\rho(f)))(x) = f(x)$$

for all $f \in \mathcal{H}(F)$. From equations (4.8) and (3.36) we conclude that there are two maps, namely, $\sigma : \mathcal{H}_E(\mathcal{P}) \rightarrow \mathcal{H}(F)$ and $\rho : \mathcal{H}(F) \rightarrow \mathcal{H}_E(\mathcal{P})$ such that $\rho\sigma$ is the identity map on $\mathcal{H}_E(\mathcal{P})$, and $\sigma\rho$ is the identity map on $\mathcal{H}(F)$. Therefore, we have the bijection

$$\mathcal{H}_E(\mathcal{P}) \leftrightarrow \mathcal{H}(F).$$

Combining this bijection with (3.21), we have the following bijections:

$$\mathcal{H}(E) \leftrightarrow \mathcal{H}_E(\mathcal{P}) \leftrightarrow \mathcal{H}(F), \quad (4.9)$$

which completely describes all invariant functions for F ; they are just given by “arbitrary” functions on the elementary region E for F .

By collecting the previous results we have the following theorem:

Theorem 3 *The invariant subspace of the groups which belong to the classes E_2 , E_3 , and E_4 consists just of one element, namely, the zero function. The invariant subspace of a group F which either belongs to the class O or to the class E_1 consists of functions which are “arbitrary” on the elementary region E for F . In particular, the following bijections hold*

$$\mathcal{H}(E) \leftrightarrow \mathcal{H}_E(\mathcal{P}) \leftrightarrow \mathcal{H}(F).$$

A subgroup of $C_n \times C_m$ of odd order can never be a potential little group. Indeed, let O be such a group. Then, there always exists a bigger group, namely $O \times A$, where, $A = \{(I, I), (-I, -I)\} \simeq Z_2$ which has the same invariant vectors as O . So, by taking into account theorem 3 we arrive at the following

Proposition 9 *The only finite potential little groups are those subgroups of $C_n \times C_m$ which fall into the class E_1 .*

5 All the Finite Potential Little Groups are Actual

So far, for all potential little groups F , we have given an explicit description of the invariant subspaces $\mathcal{H}(F) \subset \mathcal{H}(\mathcal{P})$; $\mathcal{H}(F)$ is in bijective correspondence with $\mathcal{H}(E)$. For most of the vectors $f \in \mathcal{H}(F)$, the little group will be precisely F . However, to be certain that any given potential little group really does occur as a little group, we must exclude the possibility that all vectors $f \in \mathcal{H}(F)$ are also invariant under a bigger group $\Theta \supset F$. So, in order to prove that F does occur as a little group it suffices to prove that there exists one $f' \in \mathcal{H}(F)$ which has no higher symmetry; there is no group $\Theta \supset F$ which leaves f' invariant. The proof will be constructive. For each F we will give an $f_F \in \mathcal{H}(F)$ which is invariant only under F .

The subgroups F which fall into the class E_1 may be taken as subgroups $F \subset C_n \times C_m$, where $\pi_1(F) = C_n$ and $\pi_2(F) = C_m$. Since the groups F contain the element $(-I, -I)$ both the numbers n and m are even. Let F_{nm} be the open rectangle

$$F_{nm} = E_n \times E_m \subset \mathcal{P} \quad (5.1)$$

where $E_n = \{\theta \in S^1 \mid 0 < \theta < 4\pi/n\}$. Let R be the open rectangle with center $(2\pi/n, 2\pi/m)$ and side lengths π/n and π/m . That is, R is the set

$$R = \left(\frac{2\pi}{n} - \frac{\pi}{2n}, \frac{2\pi}{n} + \frac{\pi}{2n}\right) \times \left(\frac{2\pi}{m} - \frac{\pi}{2m}, \frac{2\pi}{m} + \frac{\pi}{2m}\right). \quad (5.2)$$

Now consider the non-effective action $\mathcal{P} \times (C_n \times C_m) \rightarrow \mathcal{P}$ defined by equation (3.10). Let R_{ij} the translate of R by the element (g_i, g_j) of $C_n \times C_m$:

$$R_{ij} = R(g_i, g_j). \quad (5.3)$$

More generally, let $R_{\omega\xi}$ denote the translate of R by any element $(g(\omega), g(\xi)) \in K$; $K = SO(2) \times SO(2)$.

$$R_{\omega\xi} = R(g(\omega), g(\xi)). \quad (5.4)$$

Then we have the following Propositions (10), (11), and (12), regarding $R_{\omega\xi}$. The proofs of these Propositions follow closely the proofs of the Propositions 4, 5, and 6 given in [2], and as such are omitted.

Proposition 10 *Let $R_{\omega\xi}$ be any translate of R . Then $R_{\omega\xi}$ cannot intersect two distinct rectangles of the form R_{ij} .*

Now let F be any finite subgroup of K which falls into the class of subgroups E_1 , with $\pi_1(F) = C_n$ and $\pi_2(F) = C_m$. So $F \subset C_n \times C_m$. Write, for convenience, $G = F$, and denote the elements of G as follows:

$$G = \{g_1, g_2, g_3, \dots, g_l\} \quad (5.5)$$

where $g_1 = e$ and l is the order of G . Let A denote the union

$$A = Rg_1 \cup Rg_2 \cup \dots \cup Rg_l \equiv RG, \quad (5.6)$$

and let B denote the union

$$B = R_{\omega\xi}g_1 \cup R_{\omega\xi}g_2 \cup \dots \cup R_{\omega\xi}g_l = R_{\omega\xi}G. \quad (5.7)$$

Then we have the following:

Proposition 11 *Let A and B be the sets defined by (5.6) and (5.7). Then either no set of the form $R_{\omega\xi}g_i$ intersects any set of the form Rg_j , or every set of the form $R_{\omega\xi}g_i$ intersects exactly two identical sets of the form Rg_j . In the latter case, we have, for some k , the union*

$$A \cap B = \cup_{j=1}^l (R_{\omega\xi} \cap Rg_k)g_j. \quad (5.8)$$

Now, for any measurable subset S of \mathcal{P} , let $\alpha(S)$ denote the area of S . Note that, for any (ω, ξ) , we have $\alpha(R_{\omega\xi}) = \alpha(R)$; the area is invariant under all translations. Now we have the following result.

Proposition 12 *Let $A = RG$ and $B = R_{\omega\xi}G$ be the sets given by (5.6) and (5.7), and suppose that*

$$\alpha(A) = \alpha(A \cap B). \quad (5.9)$$

Then $R_{\omega\xi} = Rg_k$, the sets A and B coincide, and either $(g(\omega), g(\xi)) = g_k$ or $(g(\omega), g(\xi)) = g_k(I, I)$.

We are now ready to construct a function $\zeta_3 \in \mathcal{H}(G)$ with little group G . Let χ_R be the characteristic function of R , and define ζ_3 by

$$\zeta_3 = \frac{1}{|K_F|} \sum_{g \in G} T'(g^{-1})\chi_R. \quad (5.10)$$

By construction, (as in the proof of Theorem 2, (Eq. (3.28)), $\zeta_3 \in \mathcal{H}(G)$. Recall that we denote by $\gamma(x, g)$ the multiplier $k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y)$ in the representation $(T'(g)f)(x) = \gamma(x, g)f(xg)$, where x denotes a point of the torus and g an element of $\text{SL}(2, R) \times \text{SL}(2, R)$. Henceforth, for notational convenience we will write $\gamma_g(x)$ instead of $\gamma(g, x)$.

Since for $g \in G$ we have $T'(g^{-1})\chi_R = \gamma_{g^{-1}} \cdot \chi_{Rg}$, we obtain

$$\zeta_3 = \frac{1}{|K_F|} \sum_{g \in G} \gamma_{g^{-1}} \cdot \chi_{Rg}. \quad (5.11)$$

The \cdot in $\gamma_{g^{-1}} \cdot \chi_{Rg}$ denotes pointwise multiplication. Now note that if $h^{-1} = (g(\omega)^{-1}, g(\xi)^{-1})$ is an element of $SO(2) \times SO(2)$, then $T'(g(\omega)^{-1}, g(\xi)^{-1})\zeta_3$ evaluated at $x = (\rho, \sigma)$ gives

$$\begin{aligned} (T'(g(\omega)^{-1}, g(\xi)^{-1})\zeta_3)(\rho, \sigma) &= \frac{1}{|K_F|} \sum_{g \in G} (T'(h^{-1})(\gamma_{g^{-1}} \cdot \chi_{Rg}))(\rho, \sigma) \\ &= \frac{1}{|K_F|} \sum_{g \in G} \{ \gamma_{h^{-1}}(\rho, \sigma) \gamma_{g^{-1}}(\rho - 2\omega, \sigma - 2\xi) \\ &\quad \gamma_{Rgh}(\rho, \sigma) \} \end{aligned} \quad (5.12)$$

A simple calculation shows

$$\gamma_{g^{-1}}(\rho - 2\omega, \sigma - 2\xi) = \gamma_{g^{-1}h^{-1}}(\rho, \sigma) \gamma_{h^{-1}}(\rho, \sigma).$$

Substituting back into Eq. (5.12) gives

$$T'(h^{-1})\zeta_3 = \frac{1}{|K_F|} \sum_{g \in G} \gamma_{g^{-1}h^{-1}} \cdot \chi_{Rgh}. \quad (5.13)$$

We are now ready to prove the following:

Theorem 4 *The little group of ζ_3 is precisely G ; $L(\zeta_3) = G$.*

Proof We must prove that, if $T'(g(\omega)^{-1}, g(\xi)^{-1})\zeta_3 = \zeta_3$, then

$$(g(\omega)^{-1}, g(\xi)^{-1}) \in G. \quad (5.14)$$

The equation holds in the Hilbert space sense, and is just

$$\sum_{g \in G} \gamma_{g^{-1}h^{-1}} \cdot \chi_{Rgh} = \sum_{g \in G} \gamma_{g^{-1}} \cdot \chi_{Rg}.$$

So we have

$$0 = \|\zeta_3 - T'(h^{-1})\zeta_3\|^2 = \int_{\mathbb{T}^2} \left[\sum_{g \in G} (\gamma_{g^{-1}} \cdot \chi_{Rg})(x) - \sum_{g \in G} (\gamma_{g^{-1}h^{-1}} \cdot \chi_{Rgh})(x) \right]^2 d\mu(x). \quad (5.15)$$

By using the expansion formula for $(a_1 + a_2 + a_3 + a_4 + \dots + a_\nu)^2$, where a_1, a_2, \dots, a_ν are any real numbers, and the equalities $\chi_E^2(x) = \chi_E(x)$ and $\gamma_g^2 = 1$, where $g \in G \subset C_n \times C_m$, Eq. (5.15) gives

$$0 = \int_{\mathbb{T}^2} \left\{ \sum_{g \in G} \chi_{Rg}(x) + \sum_{g \in G} \chi_{Rgh}(x) + \mathcal{B} + \mathcal{C} - \Lambda \right\} d\mu(x), \quad (5.16)$$

where, denoting by l the order of the group G , we have

$$\begin{aligned} \mathcal{B} = & 2(\gamma_{g_1^{-1}} \cdot \chi_{Rg_1})(x) \sum_{i \geq 2}^l (\gamma_{g_i^{-1}} \cdot \chi_{Rg_i})(x) + \\ & 2(\gamma_{g_2^{-1}} \cdot \chi_{Rg_2})(x) \sum_{i \geq 3}^l (\gamma_{g_i^{-1}} \cdot \chi_{Rg_i})(x) + \dots + 2(\gamma_{g_{l-1}^{-1}} \cdot \chi_{Rg_{l-1}})(x)(\gamma_{g_l^{-1}} \cdot \chi_{Rg_l})(x), \end{aligned} \quad (5.17)$$

$$\begin{aligned} \mathcal{C} = & 2(\gamma_{g_1^{-1}h^{-1}} \cdot \chi_{Rg_1h})(x) \sum_{i \geq 2}^l (\gamma_{g_i^{-1}h^{-1}} \cdot \chi_{Rg_ih})(x) + \\ & 2(\gamma_{g_2^{-1}h^{-1}} \cdot \chi_{Rg_2h})(x) \sum_{i \geq 3}^l (\gamma_{g_i^{-1}h^{-1}} \cdot \chi_{Rg_ih})(x) + \dots + \\ & 2(\gamma_{g_{l-1}^{-1}h^{-1}} \cdot \chi_{Rg_{l-1}h})(x)(\gamma_{g_l^{-1}h^{-1}} \cdot \chi_{Rg_lh})(x), \end{aligned} \quad (5.18)$$

$$\Lambda = 2 \sum_{g_i \in G} \left\{ (\gamma_{g_i^{-1}} \cdot \chi_{Rg_i})(x) \left\{ \sum_{g_j \in G} (\gamma_{g_j^{-1}h^{-1}} \cdot \chi_{Rg_jh})(x) \right\} \right\}. \quad (5.19)$$

For convenience, call $\iota = (-I, -I)$, and enumerate the elements of G as follows

$$g_1 = e, \quad g_2, \quad g_3, \quad \dots, \quad g_{\frac{l}{2}+1} = \iota, \quad g_{\frac{l}{2}+2} = g_2\iota, \quad g_{\frac{l}{2}+3} = g_3\iota, \quad \dots, \quad g_l = g_{\frac{l}{2}}\iota.$$

One then can show that

$$\mathcal{B} = 2 \sum_{i=1}^{l/2} \chi_{Rg_i}(x), \quad \mathcal{C} = 2 \sum_{i=1}^{l/2} \chi_{Rg_ih}(x). \quad (5.20)$$

Combining Eq. (5.20) with

$$\sum_{g \in G} \chi_{Rg}(x) = 2 \sum_{i=1}^{l/2} \chi_{Rg_i}(x), \quad \sum_{g \in G} \chi_{Rgh}(x) = 2 \sum_{i=1}^{l/2} \chi_{Rg_ih}(x),$$

and substituting back into Eq. (5.16) we obtain

$$0 = \int_{T^2} \left(4 \sum_{i=1}^{l/2} \chi_{Rg_i}(x) + 4 \sum_{i=1}^{l/2} \chi_{Rg_ih}(x) - \Lambda \right) d\mu(x). \quad (5.21)$$

Noting that $\sum_{i=1}^{l/2} \chi_{Rg_i}(x) = \chi_A(x)$ and that $\sum_{i=1}^{l/2} \chi_{Rg_ih}(x) = \chi_B(x)$, where A and B are the areas which are displayed in equations (5.6) and (5.7) respectively, Eq. (5.21) gives

$$0 = 4 \int_{T^2} (\chi_A(x) + \chi_B(x)) d\mu(x) - \int_{T^2} \Lambda d\mu(x). \quad (5.22)$$

Define A' and B' to be the disjoint sets $A' = A - (A \cap B)$ and $B' = B - (A \cap B)$. By noting that $\chi_A = \chi_{A'} + \chi_{A \cap B}$ and that $\chi_B = \chi_{B'} + \chi_{A \cap B}$ Eq. (5.22) gives

$$0 = 4\alpha(A') + 4\alpha(B') + 8\alpha(A \cap B) - \int_{T^2} \Lambda d\mu(x), \quad (5.23)$$

where $\alpha(A)$ denotes the area of the region A . We distinguish now the following 2 cases

1. $\alpha(A \cap B) = 0$. In this case we have $\int_{T^2} \Lambda d\mu(x) = 0$ and Eq. (5.23) gives

$$0 = \alpha(A') + \alpha(B'). \quad (5.24)$$

Since areas are always non negative, it follows that $\alpha(A') = \alpha(B') = 0$. Since $A = A' \cup (A \cap B)$ (disjoint union), we have $\alpha(A) = \alpha(A') + \alpha(A \cap B)$ and therefore we obtain $\alpha(A) = 0$. But $\alpha(A) = \frac{l}{2}\alpha(R) > 0$. Contradiction. So we cannot have $\alpha(A \cap B) = 0$. When $\alpha(A \cap B) > 0$ from Proposition 11 we have that $\alpha(R_h \cap Rg_k) > 0$, for some $k \in \{1, 2, 3, \dots, \frac{l}{2}\}$ ($R_h \equiv R_{\omega\xi}$). When $\alpha(R_h \cap Rg_k) > 0$ either $(R_h \cap Rg_k \neq \emptyset$ and $R_h \neq Rg_k$) or $R_h = Rg_k$. Now we show that the first alternative cannot happen.

2. $\alpha(A \cap B) > 0$ and $(R_h \cap Rg_k \neq \emptyset, R_h \neq Rg_k)$. In this case we have $\int_{T^2} \Lambda d\mu(x) < 8\alpha(A \cap B)$ and Eq. (5.23) gives

$$\alpha(A') + \alpha(B') < 0. \quad (5.25)$$

Contradiction, since areas are always non negative. So we cannot have $\alpha(A \cap B) > 0$ and $(R_h \cap Rg_k \neq \emptyset, R_h \neq Rg_k)$. The only remaining possibility is $\alpha(A \cap B) > 0$

and $R_h = Rg_k$. Then $A = B$ and either $h = (g(\omega), g(\xi)) = g_k$ or $h = (g(\omega), g(\xi)) = g_k(-I, -I)$. This completes the proof.

It is worth pointing out that that Equations $R_h = Rg_k$ and $0 = 4\alpha(A') + 4\alpha(B') + 8\alpha(A \cap B) - \int_{T^2} \Lambda d\mu(x)$ are mutually consistent. Indeed, when $R_h = Rg_k$, $\int_{T^2} \Lambda d\mu(x) = 8\alpha(A \cap B)$. Substituting back into Eq. (5.23) we obtain $\alpha(A') = \alpha(B') = 0$. The last Equation combined with $\alpha(A) = \alpha(A') + \alpha(A \cap B)$ gives $\alpha(A) = \alpha(A \cap B)$. Then according Proposition 12 $R_h = Rg_k$.

The previous results on little groups can be summarized as follows

Theorem 5 *The actual little groups for \mathcal{HB} are either one dimensional or zero dimensional. The infinite one dimensional such groups are the groups $H(N, p, q)$, where, N , p , and q , are all odd, and, p and q are relatively prime. The finite zero dimensional actual little groups are the groups which fall into the class E_1 .*

6 Description of the Finite Zero Dimensional Actual Little Groups

The finite zero dimensional subgroups of $SO(2) \times SO(2)$ have either one or two generators and have been given in detail in [3]. We will use this information to describe in detail the finite zero dimensional actual little groups.

6.1 Cyclic Actual Little Groups

Firstly we find the actual cyclic little groups. The result is given in Proposition 14. For this purpose we recall some relevant results, Proposition 13 and Theorem 6, proved in [3]. The element $(-I, -I)$ can only be contained in the cyclic subgroups of $C_{2^a} \times C_{2^b}$, and the actual cyclic little groups, the cyclic subgroups of $C_n \times C_m$ which fall into the class E_1 , are precisely those which contain the element $(-I, -I)$; the group D is not cyclic. The key result in [3] we need to give an explicit description of the finite cyclic actual little groups is the following: The *only* cyclic subgroups of $C_{2^a} \times C_{2^b}$ which contain the element $(-I, -I)$ are the groups $(R((\frac{2\pi}{2^k}r) i_1), R(\frac{2\pi}{2^k} i_1))$, where $1 \leq k \leq \min(a, b)$, r parametrises the groups and takes values in the set $\{1, 2, \dots, 2^k - 1\} - \{2, 2 \cdot 2, \dots, (2^{k-1} - 1)2\}$. Now we

are ready to give an explicit description of the finite cyclic actual little groups. Firstly we recall some useful results, Proposition 13 and Theorem 6, proved in [3].

Proposition 13 *Let $C_n \times C_m$ be the direct product of the cyclic groups of finite order C_n and C_m . Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ and $m = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}$ be the prime decomposition of the integers n and m , i.e., p_i , $i=1,2,\dots,s$, are distinct prime numbers and a_i , β_i are non-negative integers. Then we have the following*

1. A group

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \dots \times A_s, \quad (6.1)$$

where A_i is a cyclic subgroup, not necessarily different from the identity element, of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i=1,2,\dots,s$, is a cyclic subgroup of $C_n \times C_m$.

2. Every cyclic subgroup \mathcal{C} of $C_n \times C_m$ is of the form

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \dots \times A_s,$$

where A_i is a cyclic subgroup, not necessarily different from the identity element, of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i=1,2,\dots,s$.

3. For every cyclic group \mathcal{C} of $C_n \times C_m$ the expression (6.1) is unique.

A generator of the cyclic subgroup \mathcal{C} is given by

$$(x^{\mathcal{A}}, y^{\mathcal{B}}), \quad (6.2)$$

where, x is a generator of C_n , y is a generator of C_m ,

$$\begin{aligned} \mathcal{A} = & \sum_{i=1}^{\nu} r_i p_i^{a_i - k_i} (n/p_i^{a_i}) + \sum_{i=\nu+1}^{\nu+\chi} p_i^{a_i - k_i} (n/p_i^{a_i}) + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} j_i (n/p_i^{a_i}) + \\ & \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} p_i^{a_i - k_i} (n/p_i^{a_i}), \quad \text{and} \end{aligned} \quad (6.3)$$

$$\begin{aligned} \mathcal{B} = & \sum_{i=1}^{\nu} p_i^{b_i - k_i} (m/p_i^{b_i}) + \sum_{i=\nu+1}^{\nu+\chi} \rho_i p_i^{b_i - k_i + 1} (m/p_i^{b_i}) + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} p_i^{b_i - k_i} (m/p_i^{b_i}) + \\ & \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} j_i (m/p_i^{b_i}). \end{aligned} \quad (6.4)$$

The order $|\mathcal{C}|$ of the group \mathcal{C} is given by

$$|\mathcal{C}| = \prod_{i=1}^{\nu+\chi+\tau+\psi} p_i^{k_i}. \quad (6.5)$$

The non-negative integers ν, χ, τ, ψ are such that $\nu + \chi + \tau + \psi \leq s$. Moreover, $(p_i^{a_i}, p_i^{b_i}) = \mathcal{P}(p_i^{a_i}, p_i^{b_i})$, $i = 1, 2, \dots, s$, for some permutation \mathcal{P} of the s pairs of numbers $(p_1^{a_1}, p_1^{b_1}), (p_2^{a_2}, p_2^{b_2}), \dots, (p_s^{a_s}, p_s^{b_s})$. Furthermore, $r_i \in \{0, 1, 2, \dots, p_i^{k_i} - 1\}$, $i \in \{1, 2, \dots, \nu\}$, and, $j_\sigma \in \{0, 1, 2, \dots, p_\sigma^{a_\sigma} - 1\}$, $a_\sigma < k_\sigma \leq b_\sigma$, $\sigma \in \{\nu + \chi + 1, \nu + \chi + 2, \dots, \nu + \chi + \tau\}$. Finally, $\rho_q \in \{0, 1, 2, \dots, p_q^{k_q-1} - 1\}$, $q \in \{\nu + 1, \nu + 2, \dots, \nu + \chi\}$, and, $j_\theta \in \{0, 1, 2, \dots, p_\theta^{b_\theta} - 1\}$, $a_\theta \geq k_\theta > b_\theta$, $\theta \in \{\nu + \chi + \tau + 1, \nu + \chi + \tau + 2, \dots, \nu + \chi + \tau + \psi\}$.

One crucial feature of the cyclic subgroups \mathcal{C} of $C_n \times C_m$ is that for every subgroup \mathcal{C} the expression (6.1) is unique. This was proved in [3] with the use of Sylow's Second Theorem. It is noteworthy that one can use instead of Sylow's Second Theorem more elementary group theory in order to prove the uniqueness of (6.1). Let γ be a generator of the cyclic group $\mathcal{C} = C_{p_1^{k_1}} \times C_{p_2^{k_2}} \times C_{p_3^{k_3}} \times \dots \times C_{p_s^{k_s}}$. Then the element $\omega = \gamma^{p_2^{k_2} p_3^{k_3} \dots p_s^{k_s}}$ satisfies $\omega^{p_1^{k_1}} = (\gamma^{p_2^{k_2} p_3^{k_3} \dots p_s^{k_s}})^{p_1^{k_1}} = 1$. If the element ω had order $p_1^{m_1}$, where $m_1 < k_1$, then we would have $\gamma^{p_2^{k_2} p_3^{k_3} \dots p_s^{k_s} p_1^{m_1}} = 1$. But the order of γ equals to the order of \mathcal{C} . So we get a contradiction and therefore the order of ω is $p_1^{k_1}$. Thus ω generates $C_{p_1^{k_1}}$. Choose now any element x of order $p_1^{k_1}$. Since γ generates the whole group \mathcal{C} we have $x = \gamma^t$ for some integer t . Moreover, since x generates $C_{p_1^{k_1}}$ we have $\gamma^{tp_1^{k_1}} = 1$. Therefore, the order of γ divides $tp_1^{k_1}$, i.e., $p_1^{k_1} \dots p_s^{k_s}$ divides $tp_1^{k_1}$. Consequently, $p_2^{k_2} \dots p_s^{k_s}$ divides t . Thus $t = up_2^{k_2} \dots p_s^{k_s}$, for some integer u . We conclude that $x = \gamma^t = (\gamma^{p_2^{k_2} \dots p_s^{k_s}})^u = \omega^u$. Therefore the element x belongs to the specific copy of $C_{p_1^{k_1}}$ which is generated by ω . We conclude that for every cyclic subgroup \mathcal{C} of $C_n \times C_m$ the expression (6.1) is unique.

For the purposes of our study we also recall from [3] that a cyclic subgroup \mathcal{C} of $C_n \times C_m$ can be conveniently rewritten as a subgroup of $SO(2) \times SO(2)$.

Theorem 6 *Let n and m be any non-negative integers. Then all the finite cyclic subgroups of $SO(2) \times SO(2)$ are given by*

$$\mathcal{C} = \left(R \left(\left(\frac{2\pi}{n} \mathcal{A} \right) i \right), R \left(\left(\frac{2\pi}{m} \mathcal{B} \right) i \right) \right), \quad (6.6)$$

where the expressions \mathcal{A} and \mathcal{B} are given in (6.3) and (6.4) correspondingly. The meaning and the ranges of the parameters appearing in these expressions are displayed in Proposition 13. For each specific subgroup these parameters take specific values. Different values of the parameters correspond to different subgroups and vice versa. The order of the group \mathcal{C} is given by (6.5).

We are ready now to use Proposition 13 and Theorem 6 to describe in detail the cyclic actual little groups. Their explicit description is given in the Proposition that follows. In Theorem 7 the cyclic actual little groups are rewritten as subgroups of $SO(2) \times SO(2)$.

Proposition 14 *Let n and m be any positive even numbers. Then*

$$C_n \times C_m = (C_{2^{a_1}} \times C_{2^{\beta_1}}) \times (C_{p_2^{a_2}} \times C_{p_2^{\beta_2}}) \times (C_{p_3^{a_3}} \times C_{p_3^{\beta_3}}) \times \dots \times (C_{p_s^{a_s}} \times C_{p_s^{\beta_s}}) \quad (6.7)$$

where $a_1 \geq 1$, $\beta_1 \geq 1$, p_2, p_3, \dots, p_s are odd primes, $a_i \geq 0$, and, $\beta_i \geq 0$, $i \in \{2, 3, \dots, s\}$. Every cyclic subgroup of $K = SO(2) \times SO(2)$ which contains the element $(-I, -I)$ is written uniquely in the form

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \dots \times A_s,$$

where A_i is a cyclic subgroup, not necessarily different from the identity element, of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i \in \{2, \dots, s\}$, and A_1 , ($p_1 = 2$), is restricted to be one of the following cyclic subgroups of $C_{2^{a_1}} \times C_{2^{\beta_1}}$

$$\left(R \left(\left(\frac{2\pi}{2^{k_1}} r \right) i_1 \right), R \left(\frac{2\pi}{2^{k_1}} i_1 \right) \right),$$

where $1 \leq k_1 \leq \min(a_1, \beta_1)$, r parametrises the groups and takes values in the set $\{1, 2, \dots, 2^{k_1} - 1\} - \{2, 2 \cdot 2, \dots, (2^{k_1-1} - 1)2\}$ and i_1 enumerates the elements of each group and takes values in the set $\{0, 1, 2, \dots, 2^{k_1} - 1\}$. A generator of \mathcal{C} is given by (6.2), where, $(p_i^{a_i}, p_i^{b_i}) = \mathcal{P}(p_i^{a_i}, p_i^{\beta_i})$, $i \in \{1, 2, \dots, s\}$, for some permutation \mathcal{P} of the s pairs of numbers $(p_1^{a_1}, p_1^{\beta_1}), (p_2^{a_2}, p_2^{\beta_2}), \dots, (p_s^{a_s}, p_s^{\beta_s})$, and where, n, m are positive even numbers. In (6.2) one of the primes p_1, p_2, \dots, p_ν is the prime number 2. If say, $p_t = 2$, $t \in \{1, 2, \dots, \nu\}$, then $r_t \in \{1, 2, \dots, 2^{k_t} - 1\} - \{2, 2 \cdot 2, \dots, (2^{k_t-1} - 1)2\}$. The rest of the indices r_d , $d \in \{1, 2, \dots, \nu\} - \{t\}$, take values in the sets $\{0, 1, 2, \dots, p_d^{k_d} - 1\}$. The other indices which appear in (6.2) take values in the sets which are displayed in Proposition (13). Some, or in fact all the exponents a_i and b_i , $i \in \{1, 2, 3, \dots, s\} - \{t\}$, which appear in (6.2) can be equal to zero.

Theorem 7 *The cyclic subgroup \mathcal{C} can be written as*

$$\left(R \left(\left(\frac{2\pi}{n} \mathcal{A} \right) i \right), R \left(\left(\frac{2\pi}{m} \mathcal{B} \right) i \right) \right),$$

where \mathcal{A} and \mathcal{B} are given respectively by the expressions (6.3) and (6.4). The ranges of the parameters which appear in \mathcal{A} and \mathcal{B} are specified in Proposition (14).

6.2 Actual Little Groups with Two Generators

We describe now explicitly in Proposition 16 the finite actual little groups with two generators. The key observation here is that there is no subgroup of $C_{2^a} \times C_{2^\beta}$ with two generators which falls into the class E_1 [3]. Therefore, as in the case of the cyclic actual groups, the subgroups of $C_{2^a} \times C_{2^\beta}$ are restricted to be one of the groups $\left(R \left(\left(\frac{2\pi}{2^k} r \right) i_1 \right), R \left(\left(\frac{2\pi}{2^k} i_1 \right) \right) \right)$, where, $1 \leq k \leq \min(a, \beta)$, r parametrises the groups and takes values in the set $\{1, 2, \dots, 2^k - 1\} - \{2, 2 \cdot 2, \dots, (2^{k-1} - 1)2\}$. We need to recall first some relevant results, Proposition 15 and Theorem 8, from [3].

Proposition 15 *Let $C_n \times C_m$ be the direct product of the cyclic groups of finite order C_n and C_m . Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ and $m = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}$ be the prime decomposition of the integers n and m , i.e., p_i , $i=1, 2, \dots, s$, are distinct prime numbers and a_i , β_i are non-negative integers. Then we have the following*

1. *A group*

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \dots \times A_s \tag{6.8}$$

where A_i is a subgroup, not necessarily different from the identity element, of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i=1, 2, \dots, s$, is a subgroup of $C_n \times C_m$ with two generators if at least one of the A_i , $i=1, 2, \dots, s$, has two generators.

2. *Every subgroup \mathcal{C} of $C_n \times C_m$ with two generators is of the form*

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \dots \times A_s,$$

where A_i is a subgroup, not necessarily different from the identity element, of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i=1, 2, \dots, s$, and, where at least one of the A_i , $i=1, 2, \dots, s$, has two generators.

3. *For every subgroup \mathcal{C} of $C_n \times C_m$ with two generators the expression (6.8) is unique.*

Two generators of a subgroup \mathcal{C} of $C_n \times C_m$ with two generators are given by

1.

$$g_1 = (x^{\mathcal{A}_1}, y^{\mathcal{B}_1}), \quad (6.9)$$

where, x is a generator of C_n , y is a generator of C_m ,

$$\begin{aligned} \mathcal{A}_1 = & \sum_{i=1}^{\nu} r_i p_1^{a_i - k_i} (n/p_i^{a_i}) + \sum_{i=\nu+1}^{\nu+\chi} p_i^{a_i - k_i} (n/p_i^{a_i}) + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} j_i (n/p_i^{a_i}) + \\ & \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} p_i^{a_i - k_i} (n/p_i^{a_i}) + \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} r_i p_i^{a_i - k_i} (n/p_i^{a_i}) + \\ & \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} p_i^{a_i - k_i} (n/p_i^{a_i}) + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} t_i (n/p_i^{a_i}) + \\ & \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{a_i - k_i} (n/p_i^{a_i}) \end{aligned} \quad (6.10)$$

and

$$\begin{aligned} \mathcal{B}_1 = & \sum_{i=1}^{\nu} p_i^{b_i - k_i} (m/p_i^{b_i}) + \sum_{i=\nu+1}^{\nu+\chi} \rho_i p_i^{b_i - k_i + 1} (m/p_i^{b_i}) + \sum_{i=\nu+\chi+1}^{\nu+\chi+\tau} p_i^{b_i - k_i} (m/p_i^{b_i}) + \\ & \sum_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} j_i (m/p_i^{b_i}) + \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} p_i^{b_i - k_i} (m/p_i^{b_i}) + \\ & \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} \rho_i p_i^{b_i - k_i + 1} (m/p_i^{b_i}) + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{b_i - k_i} (m/p_i^{b_i}) + \\ & \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} t_i (m/p_i^{b_i}) \end{aligned} \quad (6.11)$$

and by

2.

$$g_2 = (x^{\mathcal{A}_2}, y^{\mathcal{B}_2}), \quad (6.12)$$

where,

$$\mathcal{A}_2 = \sum_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma} p_i^{a_i-1_i}(n/p_i^{a_i}) + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{a_i-1_i}(n/p_i^{a_i}) \quad (6.13)$$

and

$$\mathcal{B}_2 = \sum_{i=\nu+\chi+\tau+\psi+\sigma+1}^{\nu+\chi+\tau+\psi+\sigma+\theta} p_i^{b_i-1_i}(m/p_i^{b_i}) + \sum_{i=\nu+\chi+\tau+\psi+\sigma+\theta+\phi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{b_i-1_i}(m/p_i^{b_i}). \quad (6.14)$$

The order $|\mathcal{C}|$ of the group \mathcal{C} is given by

$$|\mathcal{C}| = \prod_{i=1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{k_i} \times \prod_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi} p_i^{l_i}. \quad (6.15)$$

The non-negative integers $\nu, \chi, \tau, \psi, \sigma, \theta, \phi, \xi$ are such that $\nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi \leq s$. Moreover, $(p_i^{a_i}, p_i^{b_i}) = \mathcal{P}(p_i^{a_i}, p_i^{b_i})$, $i = 1, 2, \dots, s$, for some permutation \mathcal{P} of the s pairs of numbers $(p_1^{a_1}, p_1^{b_1}), (p_2^{a_2}, p_2^{b_2}), \dots, (p_s^{a_s}, p_s^{b_s})$. Furthermore, when $i \in \{1, 2, \dots, \nu\}$, then $r_i \in \{0, 1, 2, \dots, p_i^{k_i} - 1\}$ and $k_i \leq \min(a_i, b_i)$, and when, $w \in \{\nu + \chi + 1, \nu + \chi + 2, \dots, \nu + \chi + \tau\}$, then $j_w \in \{0, 1, 2, \dots, p_w^{a_w} - 1\}$, and $a_w < k_w \leq b_w$. When $q \in \{\nu + 1, \nu + 2, \dots, \nu + \chi\}$, then $\rho_q \in \{0, 1, 2, \dots, p_q^{k_q-1} - 1\}$ and $k_q \leq \min(a_q, b_q)$, and when, $y \in \{\nu + \chi + \tau + 1, \nu + \chi + \tau + 2, \dots, \nu + \chi + \tau + \psi\}$ then $j_y \in \{0, 1, 2, \dots, p_y^{b_y} - 1\}$ and $a_y \geq k_y > b_y$. When $i_1 \in \{\nu + \chi + \tau + \psi + 1, \nu + \chi + \tau + \psi + 2, \dots, \nu + \chi + \tau + \psi + \sigma\}$, then $r_{i_1} \in \{0, 1, 2, \dots, p_{i_1}^{k_{i_1}-1_{i_1}} - 1\}$ and $1 \leq l_{i_1} \leq k_{i_1} \leq \min(a_{i_1}, b_{i_1})$, and when, $q_1 \in \{\nu + \chi + \tau + \psi + \sigma + 1, \dots, \nu + \chi + \tau + \psi + \sigma + \theta\}$, then $\rho_{q_1} \in \{0, 1, 2, \dots, p_{q_1}^{k_{q_1}-1_{q_1}} - 1\}$ and $1 \leq l_{q_1} < k_{q_1} \leq \min(a_{q_1}, b_{q_1})$. When $w_1 \in \{\nu + \chi + \tau + \psi + \sigma + \theta + 1, \nu + \chi + \tau + \psi + \sigma + \theta + 2, \dots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi\}$, then $t_{w_1} \in \{0, 1, 2, \dots, p_{w_1}^{a_{w_1}-1_{w_1}} - 1\}$ and $1 \leq l_{w_1} \leq a_{w_1} < k_{w_1} \leq b_{w_1}$. Finally, when $y_1 \in \{\nu + \chi + \tau + \psi + \sigma + \theta + \phi + 1, \nu + \chi + \tau + \psi + \sigma + \theta + \phi + 2, \dots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi\}$, then $j_{y_1} \in \{0, 1, 2, \dots, p_{y_1}^{b_{y_1}-1_{y_1}} - 1\}$ and $1 \leq l_{y_1} \leq b_{y_1} < k_{y_1} \leq a_{y_1}$.

We conclude therefore that every subgroup \mathcal{C} of $C_n \times C_m$ with two generators is the direct product of two cyclic groups \mathcal{C}_1 and \mathcal{C}_2 :

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2. \quad (6.16)$$

By construction, the order of the non-cyclic group \mathcal{C} is given by (6.15). As it was pointed out before the choice of the cyclic subgroups \mathcal{C}_1 and \mathcal{C}_2 in expression (6.16)

is highly non-unique. The choices displayed in expressions (6.9) and (6.12) are only two specific choices among the many possible. What is common in all these choices is that the orders of the cyclic groups \mathcal{C}_1 and \mathcal{C}_2 are *not* relatively prime. In fact, in the particular choice we made we have

$$|\mathcal{C}_1| = \prod_{i=1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{k_i} \quad (6.17)$$

and,

$$|\mathcal{C}_2| = \prod_{i=\nu+\chi+\tau+\psi+1}^{\nu+\chi+\tau+\psi+\sigma+\theta+\phi+\xi} p_i^{l_i}. \quad (6.18)$$

For the purposes of our study it is convenient to rewrite \mathcal{C}_1 and \mathcal{C}_2 as subgroups of $\text{SO}(2) \times \text{SO}(2)$. This is the content of the following Theorem.

Theorem 8 *Let n and m be any non-negative integers. Then all the finite non-cyclic subgroups \mathcal{C} of $\text{SO}(2) \times \text{SO}(2)$ can be written as the direct product of two cyclic groups \mathcal{C}_1 and \mathcal{C}_2 whose orders are not relatively prime. Thus we have*

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2. \quad (6.19)$$

The choice of \mathcal{C}_1 and \mathcal{C}_2 is highly non unique. A possible choice for \mathcal{C}_1 is given by

$$\mathcal{C}_1 = \left(R \left(\left(\frac{2\pi}{n} \mathcal{A}_1 \right) i \right), R \left(\left(\frac{2\pi}{m} \mathcal{B}_1 \right) i \right) \right), \quad (6.20)$$

and a possible choice for \mathcal{C}_2 is given by

$$\mathcal{C}_2 = \left(R \left(\left(\frac{2\pi}{n} \mathcal{A}_2 \right) i \right), R \left(\left(\frac{2\pi}{m} \mathcal{B}_2 \right) i \right) \right). \quad (6.21)$$

The meaning and the ranges of the parameters \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{A}_2 , \mathcal{B}_2 appearing in these expressions are displayed in Proposition (15). For each specific subgroup these parameters take specific values. Different values of the parameters correspond to different subgroups and vice versa. The order of the group \mathcal{C} is given by (6.15).

Using now Proposition 15, Theorem 8 and the observation made at the beginning of this subsection we give now in detail the little groups with two generators.

Proposition 16 *Let n and m be any positive even numbers. Then*

$$C_n \times C_m = (C_{2^{a_1}} \times C_{2^{\beta_1}}) \times (C_{p_2^{a_2}} \times C_{p_2^{\beta_2}}) \times (C_{p_3^{a_3}} \times C_{p_3^{\beta_3}}) \times \dots \times (C_{p_s^{a_s}} \times C_{p_s^{\beta_s}}) \quad (6.22)$$

where $a_1 \geq 1$, $\beta_1 \geq 1$, p_2, p_3, \dots, p_s are odd primes, $a_i \geq 0$, and, $\beta_i \geq 0$, $i \in \{2, 3, \dots, s\}$. Every subgroup \mathcal{C} of $K = SO(2) \times SO(2)$ with two generators which falls into the class E_1 is written uniquely in the form

$$\mathcal{C} = A_1 \times A_2 \times A_3 \times \dots \times A_s,$$

where A_i is a subgroup of $C_{p_i^{a_i}} \times C_{p_i^{\beta_i}}$, $i \in \{1, 2, \dots, s\}$, ($p_1 = 2$). A_1 is not the identity element and must be cyclic. In particular, A_1 is restricted to be one of the following cyclic subgroups of $C_{2^{a_1}} \times C_{2^{\beta_1}}$

$$\left(R \left(\left(\frac{2\pi}{2^{k_1}} r \right) i_1 \right), R \left(\frac{2\pi}{2^{k_1}} i_1 \right) \right),$$

where $1 \leq k_1 \leq \min(a_1, \beta_1)$, r parametrises the groups and takes values in the set $\{1, 2, \dots, 2^{k_1} - 1\} - \{2, 2 \cdot 2, \dots, (2^{k_1-1} - 1)2\}$ and i_1 enumerates the elements of each group and takes values in the set $\{0, 1, 2, \dots, 2^{k_1} - 1\}$. One of the A_i , $i \in \{2, \dots, s\}$, which are not all necessarily different from the identity element, has two generators. Two generators of \mathcal{C} are given by (6.9) and (6.12), where, $(p_i^{a_i}, p_i^{b_i}) = \mathcal{P}(p_i^{a_i}, p_i^{\beta_i})$, $i \in \{1, 2, \dots, s\}$, for some permutation \mathcal{P} of the s pairs of numbers $(p_1^{a_1}, p_1^{\beta_1}), (p_2^{a_2}, p_2^{\beta_2}), \dots, (p_s^{a_s}, p_s^{\beta_s})$, and where, n and m are positive even numbers and one of the primes p_1, p_2, \dots, p_ν is the prime number 2. If say, $p_t = 2$, $t \in \{1, 2, \dots, \nu\}$, then $r_t \in \{1, 2, \dots, 2^{k_t} - 1\} - \{2, 2 \cdot 2, \dots, (2^{k_t-1} - 1)2\}$. The rest of the indices r_d , $d \in \{1, 2, \dots, \nu\} - \{t\}$, take values in the sets $\{0, 1, 2, \dots, p_d^{k_d} - 1\}$. The other indices which appear in (6.9) and (6.12) take values in the sets which are displayed in Proposition (15). In (6.9) and (6.12) some of the exponents a_i and b_i , $i \in \{1, 2, 3, \dots, \nu + \chi + \tau + \psi\} - \{t\}$, or in fact all of them, can be equal to zero. On the other hand, in (6.9) and (6.12) at least one of the products of primes $a_i \cdot b_i \neq 0$, $i \in \{\nu + \chi + \tau + \psi + 1, \dots, \nu + \chi + \tau + \psi + \sigma + \theta + \phi + \xi\}$.

For the purposes of this study it is convenient to rewrite the little groups with two generators \mathcal{C} as subgroups of $SO(2) \times SO(2)$.

Theorem 9 *Every little group \mathcal{C} with two generators is written as a direct product of two cyclic groups in a highly non-unique way. A possible choice is the following*

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2,$$

where,

$$\mathcal{C}_1 = \left(R \left(\left(\frac{2\pi}{n} \mathcal{A}_1 \right) i_1 \right), R \left(\left(\frac{2\pi}{m} \mathcal{B}_1 \right) i_1 \right) \right), \quad (6.23)$$

and where,

$$\mathcal{C}_2 = \left(R \left(\left(\frac{2\pi}{n} \mathcal{A}_2 \right) i_2 \right), R \left(\left(\frac{2\pi}{m} \mathcal{B}_2 \right) i_2 \right) \right). \quad (6.24)$$

The coefficients \mathcal{A}_1 and \mathcal{B}_1 are given by (6.10) and (6.11) correspondingly, and the coefficients \mathcal{A}_2 and \mathcal{B}_2 are given respectively by (6.13) and (6.14). The index i_1 enumerates the elements of the group \mathcal{C}_1 and takes values in the set $\{0, 1, \dots, |\mathcal{C}_1| - 1\}$, where $|\mathcal{C}_1|$ is given by (6.17), and i_2 enumerates the elements of the group \mathcal{C}_2 and takes values in the set $\{0, 1, \dots, |\mathcal{C}_2| - 1\}$, where $|\mathcal{C}_2|$ is given by (6.18). The rest of the indices which appear in (6.23) and (6.24) are given in Proposition 16.

7 Form of the induced representations

In order to give explicitly the operators of the representations of \mathcal{HB} induced from finite little groups it is necessary to give the following information [12, 13, 14, 15, 16, 17]:

1. An irreducible unitary representation U of $L(\zeta)$ on a Hilbert space D for each $L(\zeta)$.
2. A \mathcal{G} -quasi-invariant measure μ , $\mathcal{G} = G \times G$, on each orbit $\mathcal{G}\zeta \approx \mathcal{G}/L(\zeta)$; where $L(\zeta)$ denotes the little group of the base point $\zeta \in \mathcal{H}(\mathbb{T}^2)$ of the orbit $\mathcal{G}\zeta$.

The actual little groups have been given in Theorem 5. The finite ones are those which fall into the class E_1 . These are described in detail in Propositions 14 and 16.

The information cited in 1 and 2 for each of the aforementioned groups and the corresponding orbit types is now provided.

1. The finite little groups are either cyclic or can be expressed as the direct product of two cyclic groups. The little groups which are cyclic are described in detail in Proposition 14, whereas, the little groups which are the direct product of two cyclic groups are given in detail in Proposition 16. The irreducible unitary representations U_N of the cyclic groups C_N are indexed by an integer ν which, for distinct representations, takes values in the set $\nu \in \{0, 1, 2, \dots, N - 1\}$. The number of the representations equals to the order

of the group N . Denoting them by $D^{(\nu)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$D^{(\nu)} \left(\left(R \left(\left(\frac{2\pi}{n} \mathcal{A} \right) j \right), R \left(\left(\frac{2\pi}{m} \mathcal{B} \right) j \right) \right) \right) = e^{i \frac{2\pi}{N} \nu j}, \quad (7.1)$$

where, taking into account the restrictions given in Proposition 14, \mathcal{A} and \mathcal{B} are given respectively by (6.3) and by (6.4). The order $N \equiv C$ of the group C_N is given by (6.5).

Let $C_{N_1} \times C_{N_2}$ be one of the little groups which can be expressed as the direct product of two cyclic groups C_{N_1} and C_{N_2} . The unitary irreducible representations of $C_{N_1} \times C_{N_2}$ are indexed by two integers ν_1 and ν_2 which for distinct representations take independently values in the sets $\{0, 1, 2, \dots, N_1 - 1\}$ and $\{0, 1, 2, \dots, N_2 - 1\}$ (this is justified in the remark which follows). Denoting these representations by $D^{(\nu_1, \nu_2)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$\begin{aligned} D^{(\nu_1, \nu_2)} \left(\left(R \left(\left(\frac{2\pi}{n} \mathcal{A}_1 \right) j_1 \right), R \left(\left(\frac{2\pi}{m} \mathcal{B}_1 \right) j_1 \right) \right) \right) \times \\ \left(R \left(\left(\frac{2\pi}{n} \mathcal{A}_2 \right) j_2 \right), R \left(\left(\frac{2\pi}{m} \mathcal{B}_2 \right) j_2 \right) \right) = \\ e^{i \frac{2\pi}{N_1} \nu_1 j_1} e^{i \frac{2\pi}{N_2} \nu_2 j_2}, \end{aligned} \quad (7.2)$$

where, taking into account the restrictions given in Proposition 16, \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{A}_2 , and \mathcal{B}_2 , are given respectively by (6.10), (6.11), (6.13), and (6.14). The order $N_1 \equiv C_1$ of the group $(R((\frac{2\pi}{n} \mathcal{A}_1) j_1), R((\frac{2\pi}{m} \mathcal{B}_1) j_1))$ is given by (6.17) and the order $N_2 \equiv C_2$ of the group $(R((\frac{2\pi}{n} \mathcal{A}_2) j_2), R((\frac{2\pi}{m} \mathcal{B}_2) j_2))$ is given by (6.18).

A remark now is in order regarding the unitary irreducible representations of $C_{N_1} \times C_{N_2}$. The problem we encounter in this case is the determination of the unitary irreducible representations of the direct product $A \times B$, where A and B are abelian groups, and where the unitary irreducible representations of A and B are known. The group $A \times B$ is abelian and therefore its irreducible representations are one-dimensional. Moreover, since by assumption they are complex, they operate in one complex dimension $D \approx C$. Let $U(\xi, \omega)$ be a unitary irreducible complex representation of the group $A \times B$; the parameters ξ and ω enumerate the elements of the groups A and B correspondingly, and they are either continuous or discrete depending on the groups A and B , them being either continuous or finite. Then $U(\xi, \omega)$ will have the form

$$U(\xi, \omega) = u(\xi, \omega) I, \quad (7.3)$$

where I denotes the identity operator in one complex dimension. Let ξ take a specific value $\xi = \xi_o$. Then the complex number of modulus one $u(\xi, \omega)$ takes the form

$$u(\xi_o, \omega) = \alpha(\xi_o)u_B(\omega), \quad (7.4)$$

where $\alpha(\xi_o)$ is a complex number of modulus one which is a function of the specific choice $\xi = \xi_o$ we made, and $u_B(\omega)$ is a representation (unitary, irreducible) of the group B . We repeat the same argument with ω now. So, let ω take a specific value $\omega = \omega_o$. Then $u(\xi, \omega)$ equals to

$$u(\xi, \omega_o) = u_A(\xi)\beta(\omega_o), \quad (7.5)$$

where, $\beta(\omega_o)$ is a complex number of modulus one which is a function of the specific choice $\omega = \omega_o$ we made, and $u_A(\xi)$ is a representation (unitary, irreducible) of the group A . When $\xi = \xi_o$ and $\omega = \omega_o$ the right hand side of Eqs. (7.4) and (7.5) are identical and therefore we obtain

$$u(\xi_o, \omega_o) = u_A(\xi_o)u_B(\omega_o). \quad (7.6)$$

Consequently, *all* the unitary irreducibles $u(\xi, \omega)$ of the direct product $A \times B$ have the form

$$u(\xi, \omega) = u_A(\xi)u_B(\omega). \quad (7.7)$$

It is equation (7.7) which was used in (7.2) to determine the unitary irreducibles of the non-cyclic little groups.

2. We now proceed to give the information cited in 2. Although a \mathcal{G} -quasi-invariant measure is all what is needed, a \mathcal{G} -invariant measure will be provided in all cases.

01. The orbit 01 is homeomorphic to $01 \approx (\text{SL}(2, R) \times \text{SL}(2, R))/\mathcal{C}$, where the group \mathcal{C} is either cyclic or is the direct product of two cyclic groups. The coset space $(\text{SL}(2, R) \times \text{SL}(2, R))/\mathcal{C}$ is the space of orbits of the right action $R_{\mathcal{C}}$

$$\begin{aligned} R_{\mathcal{C}} &: \text{SL}(2, R) \times \text{SL}(2, R) \longrightarrow \text{SL}(2, R) \times \text{SL}(2, R) \\ R_{\mathcal{C}}((g, h)) &:= (g, h)c \end{aligned} \quad (7.8)$$

of \mathcal{C} on $\text{SL}(2, R) \times \text{SL}(2, R)$, where $(g, h) \in \text{SL}(2, R) \times \text{SL}(2, R)$ and $c \in \mathcal{C}$. Since the group \mathcal{C} is finite and since the action (7.8) is fixed point free the coset space $(\text{SL}(2, R) \times \text{SL}(2, R))/\mathcal{C}$ inherits the measure of $\text{SL}(2, R) \times \text{SL}(2, R)$.

This completes the necessary information in order to construct representations of \mathcal{HB} induced from finite little groups. The two remarks made at the Discussion of [18] are also relevant here.

8 Discussion

By using Propositions 14 and 16 one could try to give more concrete results than those of Proposition 6. The hope is that “nice looking” connected elementary regions, as opposed to those described in Proposition 6, can be found for the finite actual little groups. It turns out that this is not such an easy task. To illustrate the difficulty involved, we firstly consider the cyclic finite actual little groups. If \mathcal{C} is one of them, then, $\mathcal{C} = \mathcal{C}_{N_1} \times \mathcal{C}_{N_2}$, where, according to Proposition 14, the relatively prime numbers N_1 and N_2 can be chosen as follows

$$N_1 = \prod_{i=1}^{\nu} p_i^{k_i} \times \prod_{i=\nu+\chi+1}^{\nu+\chi+\tau} p_i^{k_i} \quad \text{and} \quad N_2 = \prod_{i=\nu+1}^{\nu+\chi} p_i^{k_i} \times \prod_{i=\nu+\chi+\tau+1}^{\nu+\chi+\tau+\psi} p_i^{k_i},$$

where one of the primes p_1, p_2, \dots, p_ν is the number 2 and the ranges of the exponents k_i are given in Proposition 14. One can prove that an elementary region for the group \mathcal{C}_{N_1} is given by $E_{N_1} = \{(\rho, \sigma) \in P_1(R) \times P_1(R) \mid 0 \leq \rho < 2\pi, \ 0 \leq \sigma < 2\pi/(N_1/2)\}$ and an elementary region for the group \mathcal{C}_{N_2} is given by $E_{N_2} = \{(\rho, \sigma) \in P_1(R) \times P_1(R) \mid 0 \leq \rho < 2\pi/N_2, \ 0 \leq \sigma < 2\pi\}$. One is tempted to conjecture that an elementary region for the cyclic group \mathcal{C} is given by the intersection $E_{N_1} \cap E_{N_2}$. It turns out that this is wrong. In specific cases one can easily construct counterexamples where $E_{N_1} \cap E_{N_2}$ is not an elementary domain for the \mathcal{C} -action. To illustrate the subtlety of the problem a few more remarks are here in order. The definition of elementary domain which is given in section 2 is equivalent to the following one: (Def 2) Let M be any topological space, and let G be any finite group which acts on M from the right. An *elementary domain* for the given action is an open subset $E \subset M$ such that *every* G -orbit intersects E at only one point. In turn this last definition is equivalent to the following one: (Def 3) Let M be any topological space, and let G be any finite group which acts on M from the right. An *elementary domain* for the given action is an open subset $E \subset M$ which satisfies: $\forall x \in E, \quad xg \in E \Rightarrow g = I$, where I denotes the identity element of the group G . In our problem, it can be proved that the area $\mathcal{E} = E_{N_1} \cap E_{N_2}$ satisfies the following

$$\mathcal{E}g = \mathcal{E} \Rightarrow g = I. \tag{8.1}$$

Now, one can show that Def 3 implies Eq. (8.1). But Eq. (8.1) does not imply Def 3, and in fact, as we have already said the region $\mathcal{E} = E_{N_1} \cap E_{N_2}$ is *not* an elementary region for the \mathcal{C} -action on \mathcal{P} . Similar problems are encountered when one tries to find elementary regions for the finite actual little groups with two generators.

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